QUASICONFORMAL MAPPINGS ON PLANAR SURFACES

BY

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DISSERTATION

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Abstract

This thesis discusses three different projects concerning quasiconformal mappings on planar surfaces. In the first two projects we show that a priori weaker conditions still suffice to prove quasiconformality. The geometric definition states that an orientation-preserving homeomorphism $f : U \to f(U)$ is quasiconformal if there exists $K \geq 1$ such that for all Q, $\overline{Q} \subset U$ the ratio of the modulus of $f(Q)$ to the modulus of Q is bounded above by K. We show that for the subclass of homeomorphisms that preserves the set of vertical lines, it suffices to just consider squares with sides parallel to the coordinate axes and at forty-five degree angles to the coordinate axes. Another more recent sufficient condition for quasiconformality discovered by Hubbard in 2006, requires that the skews of triangles be only distorted by a bounded amount. Haïssinsky, Hinkkanen and I proved that if there exists a constant K such that $\text{skew}(f(T)) \leq K$ for all equilateral triangles T, then f is quasiconformal. Furthermore, this condition is also sufficient for mappings between finite-dimensional Hilbert spaces.

In the last project we study quasiconformal mappings on a generalized class of Grushin planes. We define quasisymmetries between these Grushin planes and the complex plane, and use them to find a Grushin Beltrami equation and state an analytic definition of quasisymmetry on the Grushin plane. Finally we look at a previously discovered class of conformal mappings on the Grushin plane, and show that these conformal mappings agree with our analytic definition of quasisymmetry in the conformal case.

To my parents who first introduced me to math.

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Lastly, I would like to thank my parents for first introducing me to math, and showing me how beautiful and fun math can be!

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Chapter 1

Notation

$\overline{AB}$, the line segment from A to B

$A_b(S)$, see 4.3

$A_t(S)$, see 4.3

a.e., except on a set of Lebesgue measure zero

$\arg(z)$, the unique number $\theta \in [0, 2\pi)$ such that $z = re^{i\theta}$ and $r > 0$

$f(t) \asymp g(t)$, for some constant C, $(1/C)f(t) \leq g(t) \leq Cf(t)$

$B(x, R) = \{y : d(x, y) < R\}$, the open ball of radius R

$B_l(S)$, see 4.3

$B_r(S)$, see 4.3

$C(z, r) = \{z' : |z - z'| = r\}$, the circle of radius r

$\overline{E}$, the closure of the set E

$\mathrm{diam}(E) = \sup\{\mathrm{dist}(x, y) : x, y \in E\}$, the diameter of the set E

$d(x, y)$, the distance between x and y in the indicated metric space

d_r, see Definition 11 and Remark 3 in 6.2

d_{rCC}, see Definition 8 in 6.1 and Remark 3 in 6.2

$D(z, r) = \{z' : |z - z'| \leq r\}$, the closed Euclidean disk of radius r.

$D_r g$, see Definition 14 in 6.5

$|x - y|$, the Euclidean distance between x and y

$d(z, E) = \inf\{d(z, z') : z' \in E\}$

$f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

$f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

G_r, see Definition 8 in 6.1 and Remark 3 in 6.2

$H(z,r) = \frac{M(z,r)}{m(z,r)}$

$\mathcal{H}^k_\epsilon(E) = \inf\{\sum_{i=1}^\infty (\mathrm{diam}(A_i))^k : \cup_{i=1}^\infty A_i \supset E,\ \mathrm{diam}(A_i) < \epsilon\}$

$\mathcal{H}^k(E) = \lim_{\epsilon\to 0} \mathcal{H}^k_\epsilon(E)$, k-dimensional Hausdorff measure

$\mathrm{Im}(z)$, the imaginary part of a complex number z

$\mathrm{Im}(A_i(S))$, the set of imaginary parts of all the points in $A_i(S)$ where $i \in \{b,t\}$

$\mathrm{Im}(B_j(S))$, the set of imaginary parts of all the points in $B_j(S)$ where $j \in \{l,r\}$

$J(A)$, the determinant of the matrix A

$L(T) = \max\{|a-b| : a, b$ are vertices of the triangle $T\}$

$l(T) = \min\{|a-b| : a, b$ are distinct vertices of the triangle $T\}$

$\ell(Q) = \mathrm{diam}\{\mathrm{Re}(z) : z \in Q\}$

$||f||_\infty = \inf\{M : |f(z)| \leq M$ a.e.$\}$

$m(Q)$, the area of a quadrilateral, Q

$M(Q)$, the modulus of a quadrilateral, Q. See discussion before Definition 4 in 3.1

$M(z,r) = \sup_{|z-z'|=r} |f(z) - f(z')|$

$m(z,r) = \inf_{|z-z'|=r} |f(z) - f(z')|$

$|z|$, the Euclidean norm of a complex number z

$\partial_\theta f(z) = \lim_{r\to 0^+} \frac{f(z+re^{i\theta})-f(z)}{r}$, the partial derivative of f at z in the direction of θ

$P(X,K)$, see the beginning of 3.3.1

$\mathrm{Re}(z)$, the real part of a complex number z

$\mathrm{Re}(A_i(S))$, the set of real parts of all the points in $A_i(S)$ where $i \in \{b,t\}$

$\mathrm{Re}(B_j(S))$, the set of real parts of all the points in $B_j(S)$ where $j \in \{l,r\}$

$s_a(Q) = \inf\{\mathrm{length}(\gamma) : \gamma$ is a curve in $\overline{Q}$ with $\gamma(0)$ and $\gamma(1)$ in opposite a-sides of $Q\}$

$s_b(Q) = \inf\{\mathrm{length}(\gamma) : \gamma$ is a curve in $\overline{Q}$ with $\gamma(0)$ and $\gamma(1)$ in opposite b-sides of $Q\}$

$\mathrm{skew}(T) = \frac{L(T)}{l(T)}$

$U = \frac{\partial}{\partial u}$ in Chapter 6

$V = r'(u)\frac{\partial}{\partial v}$ in Chapter 6

$W = \frac{1}{2}(\frac{\partial}{\partial u} - ir'(u)\frac{\partial}{\partial v})$, see Definition 9 in 6.1

$\overline{W} = \frac{1}{2}(\frac{\partial}{\partial u} + ir'(u)\frac{\partial}{\partial v})$, see Definition 9 in 6.1

Chapter 2

Introduction

Conformal analysis is an incredibly useful branch of mathematics. One can think of conformal mappings as homeomorphisms that take infinitesimal circles to infinitesimal circles. Quasiconformal mappings relax this requirement by instead mapping infinitesimal circles to infinitesimal ellipses where the ratio of the major axis to the minor axis is uniformly bounded.

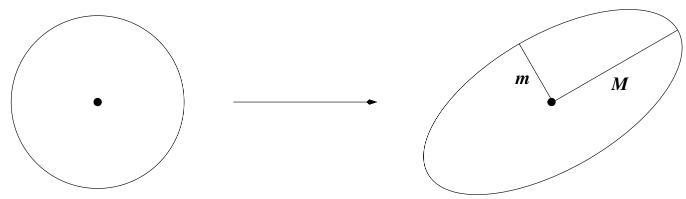

Figure 2.1: Quasiconformal mappings take infinitesimal circles to infinitesimal ellipses of uniformly bounded eccentricity, i.e., $M/m \leq K$ for some $K \geq 1$.

Quasiconformal mappings were first discovered by Grötzsch in 1928. He was looking for the mapping closest to a conformal mapping which would take a square, S, onto a rectangle, R, and map the four vertices of S onto the four vertices of R. By closest to a conformal mapping, we mean he wanted to find a differentiable homeomorphism $f : S \to R$ which minimized the quantity

$$\max\left\{\frac{\max|\partial_\theta f(z)|}{\min|\partial_\theta f(z)|} : z \in S\right\}.$$

Grötzsch showed the minimum is attained by an affine mapping [1], pp. 2-8. Today we do not require quasiconformal mappings to be differentiable. However, the analytic definition of quasiconformality has a similar flavor to Grötzsch's work.

The analytic definition of quasiconformality says we call an orientation-preserving homeomorphism K-quasiconformal if it is absolutely continuous on lines and satisfies the Beltrami equation, i.e., the ratio $f_{\bar{z}}/f_z$ exists and is equal to a measurable function μ almost everywhere such that $||\mu||_\infty$ is bounded above by $(K-1)/(K+1)$ for some $K \geq 1$. Absolute continuity on lines is a regularity condition that requires for every rectangle R in the domain of f, the restriction of f to a.e. horizontal and vertical line segment in R is absolutely continuous. A mapping is called quasiconformal if it is K-quasiconformal for some $K \geq 1$.

Quasiconformal mappings were not widely studied until in the 1930's Ahlfors and Teichmüller began using quasiconformal mappings in Nevanlinna theory and Teichmüller theory. Since then quasiconformal mappings have been applied in many areas including complex dynamics, low-dimensional topology and elliptic PDEs. For more details about these applications see [3] and [11].

One possible explanation for the wide variety of applications of quasiconformal mappings is the many flavors of characterizations of quasiconformal mappings. Another characterization of quasiconformal mappings known as the geometric definition of quasiconformality states that an orientation-preserving homeomorphism f is K-quasiconformal provided that for all topological quadrilaterals Q with closure in the domain of f, the ratio of the modulus of the quadrilateral to the modulus of its image must be uniformly bounded. The modulus of a quadrilateral is equal to the unique number $M > 0$ such that there exists a conformal mapping $\rho_Q : Q \to R_M = \{(x, y) : 0 < x < M, 0 < y < 1\}$ which is surjective, maps the vertices of Q to the vertices of R_M, and maps selected sides of Q to the horizontal sides of R_M.

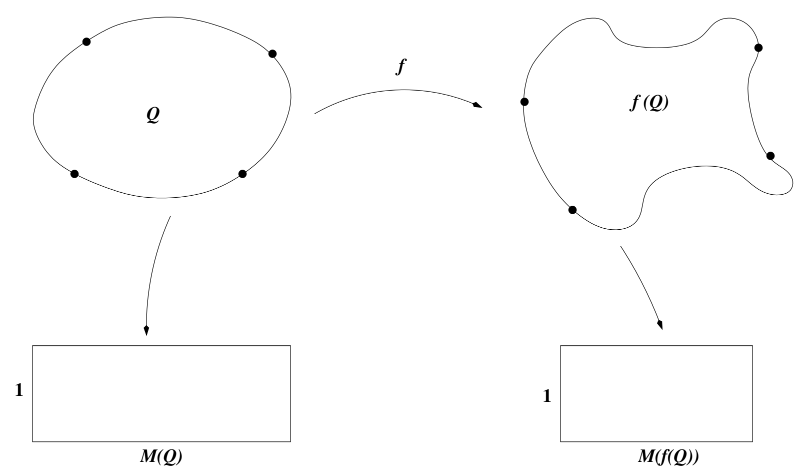

Figure 2.2: A mapping f is K-quasiconformal provided $M(f(Q))/M(Q) \leq K$.

For a reason discussed in 3.3.1 we would like to show an orientation-preserving homeomorphism is quasiconformal if for all squares, S, the modulus of $f(S)$ is uniformly bounded. In the past various results have been proved in this direction by Gehring and Väisälä [5], Palka [19] and Hinkkanen [10]. In Chapter 4 I will prove the following theorem.

Theorem 1. *Suppose U is a planar domain and $f : U \to f(U)$ is an orientation-preserving homeomorphism of the form either $f(x+iy) = g(x)+ih(x,y)$ or $f(x+iy) = g(x,y)+ih(y)$ where g and h are real-valued. If for all squares, S, with $\overline{S} \subset U$ we have*

$$M(f(S)) \leq K,$$

then f is K-quasiconformal.

Here $M(f(S))$ denotes the modulus of $f(S)$. Unfortunately I have not been able to prove Theorem 1 for all orientation-preserving homeomorphisms. Most of the difficulties arise from the extra rotational symmetry in squares.

Quasisymmetric mappings are a related class of homeomorphisms which are equivalent to quasiconformal mappings in some cases, but not all. A quasisymmetric mapping is a homeomorphism between two metric spaces with the property that for any three points z_1, z_2, z_3 the ratio $d(z_1, z_2)/d(z_1, z_3)$ is distorted only by a bounded amount.

In 2006 Hubbard gave a new characterization of planar quasiconformal mappings with a similar flavor to quasisymmetry. His characterization involved bounding the change under the mapping of skews of triangles [11]. The skew of a topological triangle is the ratio of the largest distance between any two vertices to the smallest distance between any two distinct vertices. Haïssinsky, Hinkkanen and I showed it suffices to only consider equilateral triangles.

Theorem 2. *Let $f : U \to f(U)$ be an orientation-preserving homeomorphism between planar domains. If there exists $K \geq 1$ such that*

$$\text{skew}(f(T)) \leq K$$

for all closed equilaterals triangles $T \subset U$, then f is K'-quasiconformal where K' depends only on K.

Furthermore, I was able to show Theorem 2 holds in all finite-dimensional Hilbert spaces, and I suspect there are many more metric spaces in which Theorem 2 holds. This project is discussed in greater detail in Chapter 5. I hope this work will lead to easier ways to check if a homeomorphism is quasiconformal and new applications of quasiconformal mappings. Theorem 2 could possibly also be applied to circle packing techniques used to find conformal mappings. This method involves triangulating the spaces between which one wants to find a conformal mapping, and hence having a definition of quasiconformality depending only on the distortion of triangles could be useful. For more details about circle packing and how it can be applied to map the surface of the brain to the Euclidean disc, the hyperbolic disc, and the sphere quasiconformally see [22].

The theory of quasiconformal mappings has also expanded by moving beyond Euclidean space. First it was extended to the Heisenberg group [12], then general Carnot groups [17] [18] and finally equiregular sub-Riemannian manifolds [15], and Ahlfors regular metric measure spaces [9]. In Chapter 6 I discuss a theory of quasiconformal mappings in a generalized class of Grushin planes. The (classical) Grushin plane is $\mathbb{R}^2$ with a metric defined such that as you approach the vertical axis, curves with a vertical component become increasingly longer. Any curve which approaches the vertical axis with a non-horizontal tangent has infinite length. The Grushin plane is a simple example of a space that does not have the nice regularity properties which have been used to study quasiconformal mappings in the past. It is non-Ahlfors regular and has no group structure which reflects its geometry. My generalization of the Grushin plane allows for different rates at which the vertical components of curves become longer as the vertical axis is approached.

I showed the generalized Grushin planes and the complex plane are quasisymmetrically equivalent. I then used conjugation by the quasisymmetry to find a Grushin Beltrami equation and give an analytic definition of quasisymmetry on the generalized Grushin planes. Furthermore I showed that this definition agrees, in the conformal case, with previous notions of conformal mappings on the Grushin plane from a paper by Payne [20]. I hope this work will lead to further investigations of quasiconformal mappings on the Grushin plane and an expansion of the theory of quasiconformal mappings to a wider class of metric spaces.

Chapter 3

Background

In this chapter we will begin by giving preliminary material which will be used in multiple later chapters of this dissertation. In 3.1 I will introduce the theory of quasiconformal mappings by giving several definitions, providing a few examples and stating basic properties. In 3.2 I will familiarize the reader with the related concept of quasisymmetry and state how quasisymmetry compares to quasiconformality. The remaining sections are devoted to discussing specific background material and motivation for the later chapters of this thesis. Throughout this chapter, unless stated otherwise we will assume $f : U \to f(U)$ is an orientation-preserving homeomorphism of planar domains.

3.1 Quasiconformal Mappings

As stated in the Introduction planar quasiconformal mappings are homeomorphisms that take infinitesimal circles to infinitesimal ellipses of bounded eccentricity. More precisely we have the following definition:

Definition 1 (Metric Definition of Quasiconformality). *If $D(z,r) \subset U$, define*

$$M(z,r) = \sup_{|z-z'|=r} |f(z) - f(z')|,$$

$$m(z,r) = \inf_{|z-z'|=r} |f(z) - f(z')|$$

and

$$H(z,r) = \frac{M(z,r)}{m(z,r)}.$$

Then f is K-quasiconformal provided that

$$\limsup_{r\to 0} H(z,r)$$

is uniformly bounded for all $z \in U$ and is bounded by K for almost every $z \in U$.

We refer to all mappings that are K-quasiconformal for some K as quasiconformal. Conformal mappings are 1-quasiconformal, and in general one can think of K as a measure of how far the mapping differs from a conformal mapping.

Another definition of quasiconformal mappings which appears in my work is the analytic definition. Though it is far from obvious, one can show that if a homeomorphism f is metrically quasiconformal, then it is absolutely continuous on lines.

Definition 2. *The function f is absolutely continuous on lines if for every rectangle $R = \{(x,y) : a < x < b, c < y < d\}$ with $\overline{R} \subset U$, f is absolutely continuous on a.e. interval $I_x = \{(x,y) : c < y < d\}$ and a.e. interval $I_y = \{(x,y) : a < x < b\}$.*

Since f is absolutely continuous on lines, the z and $\bar{z}$ derivatives of f exist almost everywhere. At a point $z = x + iy$ where f is differentiable we have

$$\begin{aligned}
\partial_\theta f(z) &= \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta) \\
&= (\cos(\theta) + i\sin(\theta))\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) + (\cos(\theta) - i\sin(\theta))\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\
&= e^{i\theta} f_z + e^{-i\theta} f_{\bar{z}}.
\end{aligned}$$

Hence the infinitesimal distortion along the major axis is given by $|f_z| + |f_{\bar{z}}|$, and the infinitesimal distortion along the minor axis is given by $|f_z| - |f_{\bar{z}}|$. In conclusion, if f is differentiable at z, then

$$\limsup_{R\to 0} H(z,R) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|},$$

and therefore the metric definition can be rewritten to require the ratio of the z and $\bar{z}$ derivatives be bounded uniformly away from one.

Definition 3 (Analytic Definition of Quasiconformality)**.** *Suppose f is absolutely continuous on lines and satisfies*

$$f_{\bar{z}} = \mu f_z \text{ a.e.} \tag{3.1.1}$$

where μ is some measurable function with $||\mu||_\infty \leq \frac{K-1}{K+1}$ for some $K \geq 1$. Then f is quasiconformal, and we say f satisfies the (classical) Beltrami equation (3.1.1).

For a detailed proof that the metric definition of quasiconformality implies the analytic definition the reader is referred to [13], p. 178.

The measurable Riemann mapping theorem allows us to find quasiconformal mappings f which satisfy the Beltrami equation for a given function μ. This result has many diverse applications in fields including elliptic PDEs, low dimensional topology, and Teichmüller theory. For further details about these applications see [3] and [11].

The final definition of quasiconformality which I use in my research is the geometric definition. The classical Riemann mapping theorem tells us that any two Jordan domains in the complex plane are conformally equivalent. In fact one may pick three points on the boundary of each domain, require that they map to each other in some reasonable order, and still find a conformal mapping. However, this is not the case once we select a fourth point on each boundary. Let a quadrilateral be a Jordan domain with four points selected on the boundary and labeled in counter-clockwise order as z_1, z_2, z_3 and z_4. We say two quadrilaterals are equivalent to each other if there exists a conformal mapping between them such that corresponding vertices map to one another. This relation gives a set of equivalence classes each containing exactly one rectangle in the first quadrant with height one and lower left vertex z_1 at the origin. We define the modulus of a quadrilateral Q to be the width of the rectangle of this form in the quadrilateral's equivalence class and denote this quantity by $M(Q)$.

Definition 4 (Geometric Definition of Quasiconformality)**.** *The function f is quasiconformal provided that for all quadrilaterals Q such that $\overline{Q} \subset U$, we have*

$$M(f(Q))/M(Q) \leq K.$$

Remark 1. *We do not require, a priori, a lower bound* ($1/K \leq M(f(Q))/M(Q)$) *because if a quadrilateral* Q *has vertices* z_1, z_2, z_3 *and* z_4, *and* Q' *has vertices* $z_1' = z_2$, $z_2' = z_3$, $z_3' = z_4$ *and* $z_4' = z_1$, *then* $M(Q') = 1/M(Q)$. *Thus* $1/K \leq M(f(Q))/M(Q)$ *if and only if* $M(f(Q'))/M(Q') \leq K$.

The modulus of any rectangle is given by $\frac{|z_1 - z_2|}{|z_1 - z_4|}$. It is more difficult to compute the modulus of other quadrilaterals. However, if we call the Jordan arcs between z_1 and z_2, and z_3 and z_4 a-sides and the Jordan arcs between z_1 and z_4, and z_2 and z_3 b-sides, then the modulus of our quadrilateral can be bound in terms of the ratio of the interior distance between the b-sides to the interior distance between the a-sides. This fact is stated precisely in Lemma 1.

By the definition of modulus, if f is conformal then $M(Q) = M(f(Q))$ and thus f is 1-quasiconformal. The converse is also true and is proved in [13], p. 28. We can easily see a few other basic properties of quasiconformal mappings from the geometric definition.

Proposition 1. *Let* f *and* g *be* K_1*-quasiconformal and* K_2*-quasiconformal mappings respectively. Then*

(1) f^{-1} *is* K_1*-quasiconformal, and*

(2) $f \circ g$ *is* $(K_1 K_2)$*-quasiconformal.*

Proof. Let $f : U \to f(U)$ be K_1-quasiconformal, Q be a quadrilateral with $\overline{Q} \subset V$ and with vertices z_1, z_2, z_3 and z_4 and Q' the quadrilateral consisting of the same domain as Q but with vertices $z_1' = z_2$, $z_2' = z_3$, $z_3' = z_4$ and $z_4' = z_1$. Then $M(Q') \leq K_1 M(f^{-1}(Q'))$ which by Remark 1 implies $M(f^{-1}(Q)) \leq K_1 M(Q)$.

Now let $f : U \to f(U)$ be K_1-quasiconformal, $g : V \to W$ be K_2-quasiconformal and $Q \subset U$ be a quadrilateral. Then $M(f(Q)) \leq K_1 M(Q)$ and $M((g \circ f)(Q)) \leq K_2 M(f(Q))$. Combining these inequalities gives $M((g \circ f)(Q)) \leq K_1 K_2 M(Q)$. □

Other examples of quasiconformal mappings include affine mappings and bilipschitz homeomorphisms. Radial stretch maps $f(z) = z|z|^{\alpha - 1}$, $\alpha > 0$, are K-quasiconformal for $K = \max\{\frac{1}{\alpha}, \alpha\}$.

The equivalence of the geometric definition of quasiconformality and our other two definitions is given in detail in [13]. One point of interest to us is Pfluger's argument that quasiconformal mappings are absolutely continuous on lines, which is given on p. 162. In 5.5 we parallel his proof to give an alternative completion to the proof of Theorem 6. Pfluger's proof hinges on Rengel's inequality which states for all quadrilaterals Q

$$\frac{s_b^2}{m(Q)} \leq M(Q) \leq \frac{m(Q)}{s_a^2}$$

where s_a and s_b are the lengths of the shortest curves inside $\overline{Q}$ connecting the a-and b-sides of Q respectively, and $m(Q)$ is the area of Q. Our proof depends on Proposition 2 which says the images of equilateral triangles, T, under homeomorphisms satisfying the hypotheses of Theorem 6 must contain disks of size proportional to the largest distance between the images of the vertices of T.

3.2 Quasisymmetric Mappings

The three definitions of quasiconformality that we have stated are all equivalent for mappings of planar domains. The metric definition easily makes sense in other metric spaces with very few alterations. A related class of mappings which can be studied in any metric space are quasisymmetric mappings. These are equivalent to quasiconformal mappings in some cases but not all. For this section we let (X, d_X) and (Y, d_Y) be metric spaces.

Definition 5. *A homeomorphism $f : X \to Y$ is quasisymmetric if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all triples of points $a, b, c \in X$ we have*

$$d_X(a, b) \leq t d_X(b, c) \implies d_Y(f(a), f(b)) \leq \eta(t) d_Y(f(b), f(c)). \tag{3.2.1}$$

Quasisymmetric mappings of planar domains are always quasiconformal, but the converse is not always true. Inequality (3.2.1) requires that relative distances between points are not distorted too much. Quasiconformality makes the same requirement on a local scale, but not on a global scale. For example, the conformal mapping from the unit disk to the unit disk with a slit along the positive real axis is not quasisymmetric. Like the metric definition of quasiconformality, the definition of quasisymmetry makes sense in any metric space. The interplay between quasisymmetric and quasiconformal maps will arise in Chapters 5 and 6, and will be further discussed as it pertains to these two particular situations.

Another similar class of mappings are weakly quasisymmetric mappings.

Definition 6. *A homeomorphism $f : X \to Y$ is weakly quasisymmetric if there exists a constant C such that for all triples of points $a, b, c \in X$ we have*

$$d_X(a,b) \leq d_X(b,c) \implies d_Y(f(a), f(b)) \leq C d_Y(f(b), f(c)).$$

All quasisymmetric mappings are weakly quasisymmetric. Indeed, if we set $t = 1$ in equation (3.2.1) we can see that f is weakly quasisymmetric with $C = \eta(1)$. When X and Y satisfy certain criteria it can be shown that weakly quasisymmetric mappings from X to Y are quasisymmetric. For example a weakly quasisymmetric mapping from a connected doubling space to a doubling space is quasisymmetric (Theorem 10.19 in [7]). A metric space X is doubling if there exists a constant C such that all sets of diameter d can be covered by at most C sets of diameter $d/2$. We will use the fact that the Grushin plane is doubling to prove it is quasisymmetrically equivalent to the complex plane in Chapter 6.

3.3 Definitions of Quasiconformality with Weaker Conditions

One way the theory of quasiconformal mappings has expanded is by requiring a priori weaker conditions on a homeomorphism and then demonstrating that these are equivalent to the stronger previous conditions. For example it was shown by Heinonen and Koskela that the requirement in the metric definition that

$$\limsup_{R \to 0} H(z, R) \leq K \text{ } a.e.$$

can be weakened to [8]

$$\liminf_{R \to 0} H(z, R) \leq K'(K) \text{ } a.e.$$

This has been useful in the study of complex dynamics where desirable characteristics may appear in a sequence of iterates, but not for every iterate [6], p. 395.

The next two chapters of this thesis are dedicated to proving that weaker conditions suffice in the geometric definition of quasiconformality and a definition of quasiconformality given by Hubbard in his 2006 book, *Tehichmüller theory and applications to geometry, topology, and dynamics* [11].

3.3.1 Weaker Conditions for Geometric Quasiconformality

We will say f has property $P(X, K)$ where X is a set of quadrilaterals and $K \geq 1$, if for all quadrilaterals $Q \in X$ we have $M(f(Q))/M(Q) \leq K$.

Ahlfors and Pfluger showed that the geometric definition of quasiconformality is equivalent to the analytic definition of quasiconformality. Soon after it was asked whether we need to consider all quadrilaterals in the geometric definition or if it would suffice to look at some subclass of quadrilaterals. In 1961 Gehring and Väisälä showed a certain class of rectangles is sufficient [5]:

Theorem 3. *Suppose f has property $P(R, K)$ where R is the set of all rectangles with sides parallel to the coordinate axes. Then f is K'-quasiconformal where K' depends only on K.*

Their proof involved using very thin rectangles to show f is absolutely continuous on lines. The ideal situation would be to only need to consider squares. It is shown in Lehto and Virtanen [13], pp. 176, 174, 50 that if f is absolutely continuous on lines and $M(f(S)) \leq K$ for all squares S, then f is K-quasiconformal. Gehring and Väisälä also showed in 1961 that one may not consider only squares with sides parallel to the coordinate axes. Indeed, the map

$$f(z) = (\mathrm{Re}(e^{i\pi/4}z))^3 + i\mathrm{Im}(e^{i\pi/4}z)$$

satisfies property $P(S, 1)$ where S is the set of all squares with sides parallel to the coordinate axes. However, if we consider squares with sides at a 45-degree angle to the coordinate axes, then the moduli of their images grow cubically as their side lengths increase, and thus f is not quasiconformal.

Further progress was made by Palka in 1975 when he proved the following theorem and corollary [19].

Theorem 4. *Let a be a real number greater than 1 and R_a be the set of all rectangles, R, with sides parallel to the coordinate axes such that $\overline{R} \subset U$ and $M(R) = a$. Let S be the set of all squares with*

$\overline{S} \subset U$. If f satisfies $P(S,1)$ and $P(R_a,1)$, then f is a conformal mapping.

Corollary 1. *Let a be an integer greater than 1. If f satisfies $P(R_a,1)$ where R_a is defined as in Theorem 4, then f is conformal.*

These results are a significant improvement since Palka avoided using thin rectangles which were key to Gehring's and Väisälä's proof.

In 1983 Hinkkanen proved a generalization of Palka's result for quasiconformal mappings [10].

Theorem 5. *Let R_a be defined as in Theorem 4. If f satisfies $P(R_a,K)$ for some real number $a>1$, then f is K'-quasiconformal where K' depends only on K.*

As stated in Theorem 1 of the Introduction, I proved that if f is of the form $f(x+iy) = g(x)+ih(x,y)$ or $f(x+iy) = g(x,y)+ih(y)$ where g and h are real-valued, and f satisfies $P(S,K)$ where S is defined as in Theorem 4, then f is K-quasiconformal. This proof is in Chapter 4.

3.3.2 Definitions of Quasiconformality Involving the Skew of a Triangle

In his book [11] Hubbard obtained a new characterization of quasiconformal mappings. Let T be a closed topological triangle with specified vertices, $L(T) = \max\{|a-b| : a, b \text{ are vertices of } T\}$ and $l(T) = \min\{|a-b| : a, b \text{ are distinct vertices of } T\}$. We define

$$\text{skew}(T) = \frac{L(T)}{l(T)}.$$

Note that $f(T)$ is also a topological triangle so the expression $\text{skew}(f(T))$ makes sense. Then $f : U \to f(U)$ is quasiconformal provided that there exists an increasing homeomorphism $h : [0,\infty) \to [0,\infty)$ such that

$$\text{skew}(f(T)) \leq h(\text{skew}(T))$$

for all closed Euclidean triangles $T \subset U$. In fact, Hubbard showed that it is sufficient to only consider triangles with skew bounded above by $\sqrt{7/3}$. He then asked the question of whether it suffices to only consider equilateral triangles. Progress was made on this problem in a previous paper by Aramayona and Haïssinsky [2] in which they show there exists a constant $\epsilon_0 > 0$ such that if $\epsilon \in [0, \epsilon_0)$ and

$$\text{skew}(f(T)) \leq 1 + \epsilon$$

for all equilateral triangles T, then f is K-quasiconformal where K depends only on ϵ.

As stated in the Introduction, Haïssinsky, Hinkkanen and I were able to answer Hubbard's question in the affirmative. In other words we showed Aramayona's and Haïssinsky's theorem holds for all $\epsilon \geq 0$. This proof is contained in Chapter 5.

3.4 The Grushin Plane

3.4.1 Geometry of the Grushin Plane

Though quasiconformal mappings have been studied in a wide variety of spaces, the theory has been largely unexplored for metric spaces that are non-Ahlfors regular. The simplest example of such a space is the Grushin plane.

Definition 7. *The (classical) Grushin plane G is $\mathbb{R}^2$ with the metric defined by the Carnot-Carathéodory distance*

$$d_{CC}(w, w') = \inf \ell(\gamma)$$

where the infimum is taken over all absolutely continuous, horizontal paths $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to G$ with $\gamma(0) = w$ and $\gamma(1) = w'$, and the length of γ is defined by

$$\ell(\gamma) = \ell(\gamma_1, \gamma_2) = \int_0^1 \sqrt{(\gamma_1'(s))^2 + \frac{(\gamma_2'(s))^2}{(\gamma_1(s))^2}} ds.$$

Paths on the Grushin plane are called horizontal if they have a horizontal tangent at every point where they cross the vertical axis. Throughout this section and Chapter 6 we take (u, v) to be the coordinates on the Grushin plane.

The Grushin plane is Riemannian everywhere except on the singular line $u = 0$. The metric for the Grushin plane is defined using the vector fields $\frac{\partial}{\partial u}$ and $|u|\frac{\partial}{\partial v}$ which span the entire tangent space except along the vertical axis which is sub-Riemannian by Chow's condition [4], p. 23. In 6.2

we will see easily computable estimates for the Carnot-Carathéodory distance.

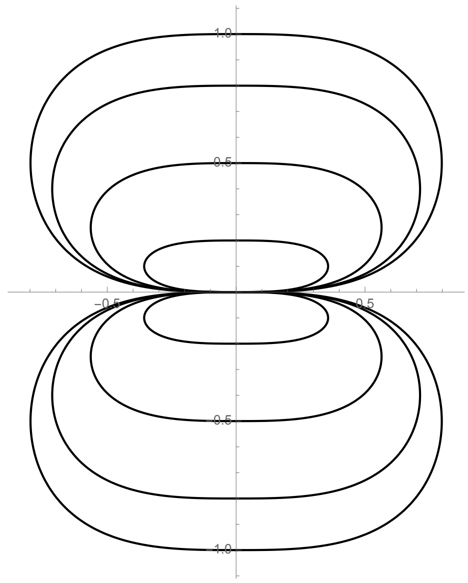

Figure 3.1: These are some geodesics starting at the origin on the Grushin plane.

When we refer to a metric space X being Ahlfors k-regular we mean that if X has Hausdorff dimension k and $\mathcal{H}^k$ is the Hausdorff k-measure on X, then there exists some constant $C \geq 1$ such that

$$\frac{1}{C}R^k \leq \mathcal{H}^k(B(x,R)) \leq CR^k$$

for all $x \in X$ and all $R > 0$ such that $B(x,R) \subset X$. The Grushin plane has Hausdorff dimension 2, and any compact subset of the Grushin plane that excludes the singular line is Ahlfors 2-regular. However, once we include the singular line in our space this fails to be true. Indeed, for small ϵ, the number of balls of radius ϵ needed to cover a disk centered at a point $(0,v)$ has magnitude $\asymp \epsilon^{-2}\ln(\epsilon^{-1})$, and thus $\mathcal{H}^2_\epsilon(B((0,v),R)) \asymp \ln(\epsilon^{-1})$ and $\mathcal{H}^2(B((0,v),R)) = \infty$ [4], p. 29.

3.4.2 Quasiconformal and Quasisymmetric Mappings on the Grushin Plane

The non-Ahlfors regularity of the Grushin plane complicates studying quasiconformal mappings on the Grushin plane because it is difficult to determine if quasisymmetry and quasiconformality are equivalent. Past theorems such as that of Heinonen and Koskela do not apply [9]. Quasisymmetry is a global condition while quasiconformality is a local one. Thus to prove that quasiconformality implies quasisymmetry we need some global regularity condition on the geometry of our space. We would like to have the equivalence of these two definitions, because it is easier to show a function satisfies the conditions for being quasiconformal, but it is usually simpler to prove theorems about quasisymmetric maps.

Meyerson showed the complex plane and the Grushin plane are quasisymmetrically equivalent via the map $(u, v) \to u|u| + iv$. He then generalized this result and showed metric spaces defined by the vector fields $\frac{\partial}{\partial u}$ and $|u|^\alpha \frac{\partial}{\partial v}$ where $\alpha > 0$, are quasisymmetrically equivalent to the complex plane [16]. In Chapter 6 we prove an even more general class of Grushin planes are quasisymmetrically equivalent to the complex plane. These quasisymmetries are of interest to us, because they can be used to translate the rich theory of quasiconformal mappings in the complex plane to the Grushin spaces via conjugation of functions.

3.4.3 Conformal Mappings on the Grushin Plane

To the best of the author's knowledge the only earlier discussion of conformal mappings on the Grushin plane prior to the work in this thesis comes from a paper by Payne [20]. He defines a sequence of flows and states that the time-s maps induced by the solutions to any of the flows are conformal maps on the Grushin plane.

Define $(\xi_k(x, y), \eta_k(x, y))$, $k \in \mathbb{N}$ by $(\xi_1, \eta_1) = (0, 1)$, $(\xi_2, \eta_2) = (2x, 4y)$, and the functions given inductively by

$$(\xi_k, \eta_k) = (2\xi_{k-1}\eta_{k-1}, \eta_{k-1}^2 - x^2\xi_{k-1}^2) \text{ for } k \geq 3.$$

The flows Payne refers to are the autonomous differential equations

$$\left(\frac{\partial x_k}{\partial s}, \frac{\partial y_k}{\partial s}\right) = (\xi_k(x_k, y_k), \eta_k(x_k, y_k))$$

where $x_k = x_k(s, u, v)$ and $y_k = y_k(s, u, v)$ are functions of u, v and a time parameter s. We will state a generalized version of these flows for our generalized Grushin planes and then solve the flows.

In our study of conformal mappings on generalized Grushin planes we will also use the definition of a conformal mapping on a Riemannian manifold. Let M be a C^∞ Riemannian manifold and g be a homeomorphism from M to M. Recall g is conformal if the pullback of the Riemannian metric by g is equal to the metric multiplied by some positive function. We will use this definition to develop a definition of conformal mappings on the complex plane. This is a reasonable approach since our generalized Grushin planes are Riemannian everywhere but the singular line.

Chapter 4

A Geometric Definition of Quasiconformality Involving Only Squares

4.1 Introduction

This chapter is devoted to proving Theorem 1 which states that if an orientation-preserving homeomorphism fixes the set of vertical lines or horizontal lines, then only squares need to be considered in the geometric definition of quasiconformality. More precisely, suppose U is a planar domain and $f : U \to f(U)$ is an orientation-preserving homeomorphism of the form either $f(x+iy) = g(x) + ih(x, y)$ or $f(x + iy) = g(x, y) + ih(y)$ where g and h are real-valued. If for all squares S such that $\overline{S} \subset U$ we have

$$M(f(S)) \leq K,$$

then f is K-quasiconformal. Please see 3.1 for the geometric definition of quasiconformality. Throughout this chapter we will assume U is a domain in the complex plane and $f : U \to f(U)$ is an orientation-preserving homeomorphism.

4.2 Proof of a Corollary to Theorem 5

I was able to prove this corollary to Theorem 5 which will be used to complete the proof of Theorem 1.

Corollary 2. *Let $R_{n,N}$ be the set of all rectangles $R = \{x + iy : x_1 < x < x_2, y_1 < y < y_2\}$ with $\overline{R} \subset U$ and $\frac{x_2 - x_1}{y_2 - y_1} \in \{n, N\}$ where n and N are any two distinct positive real numbers. If for all $R \in R_{n,N}$ we have $M(f(R)) \leq KM(R)$, then f is K'-quasiconformal where K' depends only on n, N and K.*

This is an improvement on Hinkkanen's result, because his theorem requires that $n = 1/N$.

Proof. Define $\phi : U \to \phi(U)$ by $\phi(x+iy) = \frac{1}{\sqrt{nN}}x + iy$ and consider the set of all rectangles, $R = \{x+iy : x_1 < x < x_2, y_1 < y < y_2\}$ with $\overline{R} \subset U$ and $\frac{x_2-x_1}{y_2-y_1} \in \{n, N\}$. If $\frac{x_2-x_1}{y_2-y_1} = n$, then $\frac{\phi(x_2)-\phi(x_1)}{\phi(y_2)-\phi(y_1)} = \frac{n}{\sqrt{nN}} = \sqrt{\frac{n}{N}}$. Similarly, if $\frac{x_2-x_1}{y_2-y_1} = N$, then $\frac{\phi(x_2)-\phi(x_1)}{\phi(y_2)-\phi(y_1)} = \frac{N}{\sqrt{nN}} = \sqrt{\frac{N}{n}}$. Thus ϕ maps the set of rectangles $R = \{x+iy : x_1 < x < x_2, y_1 < y < y_2\}$ with $\frac{x_2-x_1}{y_2-y_1} \in \{n, N\}$ to the set of rectangles R' with $M(R') \in \{\sqrt{\frac{N}{n}}, \sqrt{\frac{n}{N}}\}$, and therefore $M((f \circ \phi^{-1})(R')) \leq K$ for all R'. Then by Theorem 5 we have $f \circ \phi^{-1}$ is quasiconformal. Hence since ϕ is quasiconformal, we can conclude f is quasiconformal. $\square$

4.3 Notations and Conventions

Throughout the remainder of this chapter we will assume the following:

Let f be of the form

$$f(x+iy) = g(x) + ih(x,y)$$

where g and h are real-valued. This tells us that vertical lines are mapped to vertical lines. We also assume for all squares S

$$1/K \leq M(f(S)) \leq K$$

for some constant K.

When we refer to the a-side or b-side of a quadrilateral we will always consider these sets to be closed. Thus each side will contain two vertices of the quadrilateral and will have non-empty intersection with two other sides. For squares with sides parallel to the coordinate axes we will always designate the a-sides and b-sides as in Figure 4.1.

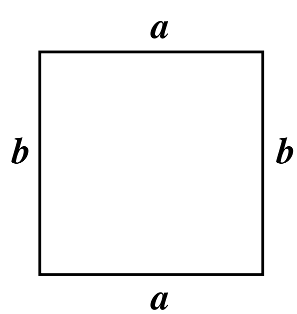

Figure 4.1: The a- and b-sides for squares with sides parallel to the coordinate axes

For squares with sides at a 45-degree angle to the coordinate axes we will designate the a- and b-sides as is shown in Figure 4.2.

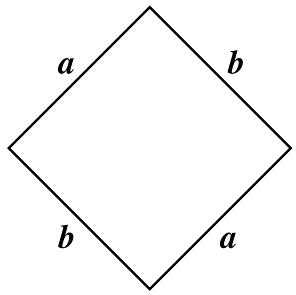

Figure 4.2: The a- and b-sides for squares with sides at a 45-degree angle to the coordinate axes

A consequence of these conventions is that in our proof we will need to bound the modulus of a class of rectangles from both above and below. This is required since we are fixing the orientation of our squares which will simplify our proofs. For more details please see the discussion just below the geometric definition of quasiconformal mappings in 3.1.

Let $B_l(S)$ be the set of points in the image under f of the left b-side of S, $B_r(S)$ be the set of points in the image under f of the right b-side of S, $A_b(S)$ be the set of points in the image under f of the bottom a-side of S and $A_t(S)$ be the set of points in the image under f of the top a-side of S. Let

(1) $\mathrm{Im}(A_i(S))$ be the set of imaginary parts of all the points in $A_i(S)$ where $i \in \{b, t\}$,

(2) $\mathrm{Re}(A_i(S))$ be the set of real parts of all the points in $A_i(S)$ where $i \in \{b, t\}$,

(3) $\mathrm{Im}(B_j(S))$ be the set of imaginary parts of all the points in $B_j(S)$ where $j \in \{l, r\}$, and

(4) $\mathrm{Re}(B_j(S))$ be the set of real parts of all the points in $B_j(S)$ where $j \in \{l, r\}$. Note for squares with sides parallel to the coordinate axes this set consists of only a single value. We will then sometimes abuse notation by thinking of $\mathrm{Re}(B_j(S))$ as a number instead of a set.

Since we take the a- and b-sides to be closed $\mathrm{Im}(A_i(S))$, $\mathrm{Re}(A_i(S))$, $\mathrm{Im}(B_j(S))$ and $\mathrm{Re}(B_j(S))$ are closed intervals.

For all quadrilaterals Q we define

$$s_b(Q) = \inf\{\text{length}(\gamma) : \gamma \text{ is a curve in } \overline{Q} \text{ with } \gamma(0) \text{ and } \gamma(1) \text{ contained in opposite } b\text{-sides of } Q\},$$

$$s_a(Q) = \inf\{\text{length}(\gamma) : \gamma \text{ is a curve in } \overline{Q} \text{ with } \gamma(0) \text{ and } \gamma(1) \text{ contained in opposite } a\text{-sides of } Q\},$$

$$\text{and } \ell(Q) = \text{diam}\{\text{Re}(z) : z \in Q\}.$$

For Lemmas 3, 4, 5 and 6, since it will be clear what S is, we will abbreviate $B_l(S)$ as B_l, $B_r(S)$ as B_r etc. The use of the argument such as S in our notation will only be used in the proofs of Lemma 2 and Theorem 1.

4.4 Proof of Theorem 1

The proof of Theorem 1 follows from Corollary 2 and the following two lemmas.

Lemma 1. *Let Q be a quadrilateral. Then*

$$\frac{1}{\pi}\frac{(\log(1+2s_b(Q)/s_a(Q)))^2}{1+2\log(1+2s_a(Q)/s_b(Q))} \leq M(Q) \leq \pi\frac{1+2\log(1+2s_b(Q)/s_a(Q))}{(\log(1+2s_a(Q)/s_b(Q)))^2}.$$

Lemma 2. *Under the assumptions of Theorem 1, there exists a constant $K' = K'(K)$ such that for all rectangles $R = \{(x,y) : x_1 < x < x_2, y_1 < y < y_2\}$ with $\overline{R} \subset U$ and $y_2 - y_1 = 2(x_2 - x_1)$, we have $1/K' \leq s_b(f(R))/s_a(f(R)) \leq K'$.*

Lemma 1 allows us to conclude that $1/K \leq M(Q) \leq K$ if and only if $1/K' \leq s_b(Q)/s_a(Q) \leq K'$ for some K' depending only on K. For a proof of Lemma 1 see [13], p. 23. Lemma 2 is proved in 4.7.

Proof of Theorem 1. By Lemmas 1 and 2 there exists a constant $K' = K'(K)$ such that for all rectangles $R = \{(x,y) : x_1 < x < x_2, y_1 < y < y_2\}$ with $\overline{R} \subset U$ and $y_2 - y_1 = 2(x_2 - x_1)$, we have

$1/K' \leq M(f(R)) \leq K'$. Then since $1/K \leq M(f(S)) \leq K$, by Corollary 2, f is quasiconformal. It is shown in [13], pp. 176, 174, 50 that we can now conclude f is K-quasiconformal. $\square$

Remark 2. *Our proof of Theorem 1 will show that to prove f is quasiconformal we only need to require that for all squares S with $\overline{S} \subset U$ and sides either parallel to the coordinate axes or at 45-degree angles to the coordinate axes, $M(f(S)) \leq K$. However, the minimal quasiconformality constant may not be K in this case which is why it is still desirable to consider all squares.*

4.5 Preliminary Lemmas

Throughout this section we assume S is a square with sides parallel to the coordinate axes and $\overline{S} \subset U$. Furthermore s_a refers to $s_a(f(S))$, s_b refers to $s_b(f(S))$, ℓ refers to $\ell(f(S))$, B_l refers to $B_l(S)$, B_r refers to $B_r(S)$, A_t refers to $A_t(S)$ and A_b refers to $A_b(S)$. We will prove several lemmas which begin to draw a relationship between s_a and s_b. We always assume the hypotheses of Theorem 1 are satisfied.

Lemma 3. *Suppose S is a square with sides parallel to the coordinate axes and $\overline{S} \subset U$. We then have $s_b \geq \ell$, and if $s_b > \ell$, then $s_a \leq \ell$ where $s_a = s_a(f(S))$, $s_b = s_b(f(S))$ and $\ell = \ell(f(S))$.*

Proof. Since f maps vertical lines to vertical lines, $\mathrm{Re}(B_l)$ and $\mathrm{Re}(B_r)$ each consist of a single value and $|\mathrm{Re}(B_l) - \mathrm{Re}(B_r)| = \ell$. We also have $s_b \geq |\mathrm{Re}(B_l) - \mathrm{Re}(B_r)|$ since the shortest length of a curve between the b-sides will be attained by a straight line segment, and any path between b-sides which is contained in $f(S)$ will be at least this long. Thus $s_b \geq \ell$.

Now suppose $s_b > \ell$. Then $f(S)$ must not contain any horizontal lines with end points on each of its b-sides. We divide our proof into two cases:

Case 1: $\mathrm{Im}(B_l) \cap \mathrm{Im}(B_r) = \emptyset$

Then the imaginary parts of all the points from one b-side must be greater than the imaginary parts of all the points from the other b-side. Without loss of generality we assume all elements of $\mathrm{Im}(B_l)$ are greater than all elements of $\mathrm{Im}(B_r)$. Now let $z_0 = B_r \cap A_t$. Since all the points in B_r must have a smaller imaginary part than $\mathrm{Im}(z_0)$ and all the points in B_l must have a larger imaginary part than $\mathrm{Im}(z_0)$, A_b must at some point intersect the line $y = \mathrm{Im}(z_0)$. Let z_b be such that $z_b \in A_b$,

$\mathrm{Im}(z_b) = \mathrm{Im}(z_0)$, and

$$\mathrm{Re}(z_b) = \max\{\mathrm{Re}(z) : \mathrm{Im}(z) = \mathrm{Im}(z_0) \text{ and } z \in A_b\}.$$

Then by the definition of ℓ, it follows that $\mathrm{Re}(A_b)$ is at most ℓ from $\mathrm{Re}(z_0)$. Therefore since $z_0 \in A_t$, and by the definition of z_b the line segment

$$\{z : \mathrm{Im}(z) = \mathrm{Im}(z_0) \text{ and } \mathrm{Re}(z_b) < \mathrm{Re}(z) < \mathrm{Re}(z_0)\} \subset f(S),$$

we have $s_a \leq \ell$.

Case 2: $\mathrm{Im}(B_l) \cap \mathrm{Im}(B_r) \neq \emptyset$

Let $A = \{z : \mathrm{Im}(z) \in \mathrm{Im}(B_l) \cap \mathrm{Im}(B_r)$ and $\mathrm{Im}(A) = \{\mathrm{Im}(z) : z \in A\}$. Then since $f(S)$ must not contain any horizontal lines from one b-side to the other, for every point in A, there must exist a point in a b-side with the same imaginary part. In other words $\mathrm{Im}(A_b)$ and $\mathrm{Im}(A_t)$ must cover $\mathrm{Im}(A)$. In fact both $\mathrm{Im}(A_t)$ and $\mathrm{Im}(A_b)$ have non-empty intersection with $\mathrm{Im}(A)$. Indeed, the maximum of $\mathrm{Im}(A)$ must be contained in $\mathrm{Im}(A_t)$ and the minimum of $\mathrm{Im}(A)$ must be contained in $\mathrm{Im}(A_b)$. Then since $\mathrm{Im}(A_t)$ and $\mathrm{Im}(A_b)$ are both compact connected sets, in order to cover $\mathrm{Im}(A)$ they must have a non-empty intersection. Thus A_t and A_b cross the same horizontal line and we therefore have, $s_a < \ell$. □

Lemma 4. *If S is a square with sides parallel to the coordinate axes and $\overline{S} \subset U$, then we have*

$$\frac{\ell}{K} \leq s_a \leq K'^2\ell \text{ and } \ell \leq s_b \leq K'\ell,$$

where K' is a constant satisfying $1/K' \leq M(f(S)) \leq K'$ for a squares S with $\overline{S} \subset U$. Recall the simplification of our notation $s_a = s_a(f(S))$, $s_b = s_b(f(S))$ and $\ell = \ell(f(S))$.

Proof. By Lemma 2 there exists a constant K' satisfying $1/K' \leq M(f(S)) \leq K'$ for all squares S with $\overline{S} \subset U$, and by Lemma 3 we can assume $s_b \geq \ell$. If $s_b = \ell$ we clearly have $\ell \leq s_b \leq K'\ell$, and

$$\frac{\ell}{K'} = \frac{s_b}{K'} \leq s_a \leq K's_b = K'\ell.$$

Now suppose $s_b > \ell$. Then $s_a \le \ell$ by Lemma 3. So $\ell < s_b \le K' s_a \le K'\ell$, and

$$\frac{\ell}{K'} < \frac{s_b}{K'} \le s_a \le K' s_b \le K'^2 \ell.$$

□

Lemma 5. *Let S be a square with sides parallel to the coordinate axes and $\overline{S} \subset U$. and let p_t and p_b be its top left and bottom right corners respectively. Then we must have*

$$\operatorname{Im}(f(p_b) - f(p_t)) \le K' \operatorname{Re}(f(p_b) - f(p_t)).$$

Proof. We may assume $\operatorname{Im}(f(p_b)) > \operatorname{Im}(f(p_t))$. Otherwise $\operatorname{Im}(f(p_b) - f(p_t)) < 0$, and since f maps vertical lines to vertical lines $\operatorname{Re}(f(p_b) - f(p_t)) > 0$ and our result is trivial. We are now in the setting of Case 1 from Lemma 3. Thus $s_a \le \ell = \operatorname{Re}(f(p_b) - f(p_t))$. Also since every element of $\operatorname{Im}(B_r)$ is greater than every element of $\operatorname{Im}(B_l)$, we have $s_b \ge \operatorname{Im}(f(p_b) - f(p_t))$. Therefore $\frac{s_b}{s_a} \le K'$ implies $\operatorname{Im}(f(p_b) - f(p_t)) \le s_b \le K' s_a \le K' \operatorname{Re}(f(p_b) - f(p_t))$. □

4.6 A Key Lemma

This lemma is exceedingly important for the proof of our theorem, because it shows that images of horizontal line segments cannot stretch over very large vertical distances without also stretching over large horizontal distances.

Lemma 6. *Suppose f maps two points with the same imaginary part to two points whose imaginary parts differ by m and real parts differ by λ. Then $m \le 3K'\lambda$.*

Proof. Let $\eta, \xi \in \mathbb{C}$ satisfy the hypotheses of our lemma. Without loss of generality we may assume $\operatorname{Re}(\xi) < \operatorname{Re}(\eta)$ and $\operatorname{Im}(f(\xi)) < \operatorname{Im}(f(\eta))$. Let D be the square with sides at 45-degree angles to the coordinate axes and left and right vertices at ξ and η respectively. Let the top vertex be denoted by α and the bottom vertex be denoted by β. We also will use the notation $f(\xi) = \xi'$, $f(\eta) = \eta'$ etc. We designate the a- and b-sides of D as stated in 4.3 and abbreviate $A_b(D)$ by A_b, $A_t(D)$ by

A_t, $B_l(D)$ by B_l and $B_r(D)$ by B_r. Throughout our proof we will use the fact that since α is above β on the same vertical line, $\mathrm{Im}(\alpha') > \mathrm{Im}(\beta')$.

Claim 1: There exist $\theta \in B_r$ and $\gamma \in B_l$ such that $|\mathrm{Im}(\theta - \gamma)| \leq K'\lambda$.

First we show $\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t) \neq \emptyset$. If $\mathrm{Im}(\alpha') > \mathrm{Im}(\eta')$, then $\mathrm{Im}(\eta') \in \mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$. This is true because $\mathrm{Im}(\xi') < \mathrm{Im}(\eta')$. So since $\mathrm{Im}(A_t)$ must contain all values between, $\mathrm{Im}(\xi')$ and $\mathrm{Im}(\alpha')$ it must contain $\mathrm{Im}(\eta')$. By definition $\mathrm{Im}(\eta') \in \mathrm{Im}(A_b)$ and thus $\mathrm{Im}(\eta') \in \mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$. Likewise if $\mathrm{Im}(\beta') < \mathrm{Im}(\xi')$, then $\mathrm{Im}(\xi') \in \mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$. Therefore the only case remaining is when we have $\mathrm{Im}(\xi') < \mathrm{Im}(\beta') < \mathrm{Im}(\alpha') < \mathrm{Im}(\eta')$. Then $\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t) \supseteq [\mathrm{Im}(\beta'), \mathrm{Im}(\alpha')]$. Thus in all cases we have $\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t) \neq \emptyset$.

Then if $[\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)] - [\mathrm{Im}(B_l) \cup \mathrm{Im}(B_r)] \neq \emptyset$ there exists a horizontal line from one a-side to the other that stays inside $f(D)$ and thus by the definition of λ we have $s_a(f(D)) \leq \lambda$. Therefore since $K' \geq \frac{s_b(f(D))}{s_a(f(D))} \geq \frac{s_b(f(D))}{\lambda}$, we have $s_b(f(D)) \leq K'\lambda$. Hence we can find θ and γ with the desired property.

Now suppose $[\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)] - [\mathrm{Im}(B_l) \cup \mathrm{Im}(B_r)] = \emptyset$. Then $\mathrm{Im}(B_l)$ and $\mathrm{Im}(B_r)$ must cover $\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$.

Subclaim a: $\mathrm{Im}(B_r) \cap [\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)] \neq \emptyset$

We showed in the first paragraph of the proof of our claim that if $\mathrm{Im}(\alpha') > \mathrm{Im}(\eta')$ then $\mathrm{Im}(\eta') \in \mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$. If $\mathrm{Im}(\alpha') < \mathrm{Im}(\eta')$, then $\mathrm{Im}(\alpha') \in \mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$. This is because $\mathrm{Im}(\beta') < \mathrm{Im}(\alpha')$ and $\mathrm{Im}(A_b)$ must contain $[\mathrm{Im}(\beta'), \mathrm{Im}(\eta')]$.

Subclaim b: $\mathrm{Im}(B_l) \cap [\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)] \neq \emptyset$

If $\mathrm{Im}(\beta') < \mathrm{Im}(\xi')$ then $\mathrm{Im}(\xi') \in \mathrm{Im}(A_b) \cap \mathrm{Im}(A_t))$ by the first paragraph of the proof of our claim. If $\mathrm{Im}(\beta') > \mathrm{Im}(\xi')$, then $\mathrm{Im}(\beta') \in \mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$. This is because $\mathrm{Im}(\beta') < \mathrm{Im}(\alpha')$ and $\mathrm{Im}(A_t)$ must contain $[\mathrm{Im}(\xi'), \mathrm{Im}(\alpha')]$.

Therefore $\mathrm{Im}(B_r)$ and $\mathrm{Im}(B_l)$ cover $\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$ and each have a non-empty intersection with $\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$. Thus since $\mathrm{Im}(A_b) \cap \mathrm{Im}(A_t)$ is compact and connected, $\mathrm{Im}(B_r) \cap \mathrm{Im}(B_l) \neq \emptyset$ and we can find θ and γ with the desired properties. This proves Claim 1.

We now have the result when $\mathrm{Im}(\gamma) < \mathrm{Im}(\xi')$ and $\mathrm{Im}(\theta) > \mathrm{Im}(\eta')$.

Suppose this is not the case. We apply Lemma 5 twice. If we take $p_t = f^{-1}(\theta)$ and $p_b = \eta$ we obtain

$$\mathrm{Im}(\eta' - \theta) \leq K'\mathrm{Re}(\eta' - \theta) \leq K'\lambda$$

and if we take $p_t = \xi$ and $p_b = f^{-1}(\gamma)$, we obtain

$$\mathrm{Im}(\gamma - \xi') \leq K'\mathrm{Re}(\gamma - \xi') \leq K'\lambda.$$

Thus if $\mathrm{Im}(\theta) < \mathrm{Im}(\eta')$ and $\mathrm{Im}(\gamma) > \mathrm{Im}(\xi')$, we have

$$m = \mathrm{Im}(\eta' - \xi') \leq \mathrm{Im}(\eta' - \theta) + |\mathrm{Im}(\theta - \gamma)| + \mathrm{Im}(\gamma - \xi') \leq 3K'\lambda.$$

If $\mathrm{Im}(\theta) > \mathrm{Im}(\eta')$ and $\mathrm{Im}(\gamma) > \mathrm{Im}(\xi')$, then

$$m \leq |\mathrm{Im}(\theta - \gamma)| + \mathrm{Im}(\gamma - \xi') \leq 2K'\lambda.$$

Finally if $\mathrm{Im}(\theta) < \mathrm{Im}(\eta')$ and $\mathrm{Im}(\gamma) < \mathrm{Im}(\xi')$, then

$$m \leq |\mathrm{Im}(\theta - \gamma)| + \mathrm{Im}(\eta' - \theta) \leq 2K'\lambda.$$

□

4.7 Proof of Lemma 2

Throughout this section we let R be a rectangle composed of two squares S_1 and S_2 such that S_1 is above S_2. More precisely we let $R = \{(x, y) : x_1 < x < x_2, y_1 < y < y_2\}$ be a rectangle such that $\overline{R} \subset U$ and $y_2 - y_1 = 2(x_2 - x_1)$. We let S_1 be the square $\{(x, y) : x_1 < x < x_2, y_1 + (y_2 - y_1)/2 < y < y_2\}$ and S_2 be the square $\{(x, y) : x_1 < x < x_2, y_1 < y < y_1 + (y_2 - y_1)/2\}$. As with squares, we let the horizontal sides of R be a-sides and the vertical sides be b-sides.

Proof of Lemma 2. We assume R is a vertical rectangle composed of the two squares S_1 and S_2 with S_1 being above S_2. Without loss of generality we assume $s_a(f(S_1)) \leq s_a(f(S_2))$. Also note in this case $\ell(f(S_1)) = \ell(f(S_2)) = \ell(f(R))$ so we will just write ℓ to denote all of these.

First we will obtain an upper bound on $s_a(f(R))$. Consider any two points on $\overline{S_1} \cap \overline{S_2}$. They are on a single horizontal line so by the definition of ℓ they can map to points with real parts differing by at most ℓ. Thus by Lemma 6 their imaginary parts can differ by at most $3K'\ell$. Therefore

$$s_a(f(R)) \leq s_a(f(S_1)) + s_a(f(S_2)) + \mathrm{diam}(f(S_1 \cap S_2)) \leq s_a(f(S_1)) + s_a(f(S_2)) + 3K'\ell + \ell.$$

Then

$$\frac{s_a(f(R))}{s_b(f(R))} \leq \frac{s_a(f(S_1)) + s_a(f(S_2)) + 3K'\ell + \ell}{\ell}.$$

So by Lemma 4

$$\frac{s_a(f(R))}{s_b(f(R))} \leq \frac{2\ell K'^2 + 3K'\ell + \ell}{\ell} = 2K'^2 + 3K' + 1.$$

Note $s_a(f(R)) \geq s_b(f(S_1)) + s_b(f(S_2))$ and $s_a(f(R)) \leq s_a(f(S_1))$. Then by Lemma 4 we have:

$$\frac{s_a(f(R))}{s_b(f(R))} \geq \frac{\frac{\ell}{K'} + \frac{\ell}{K'}}{K'\ell} = \frac{2}{K'^2}.$$

$\square$

Chapter 5

Equilateral Triangles and Quasiconformal Mappings

5.1 Introduction

In this chapter we prove Theorem 2 from Chapter 2 by proving a slightly stronger theorem.

Theorem 6. *Let U be a domain in the complex plane $\mathbb{C}$, and let $f : U \to f(U)$ be an orientation-preserving homeomorphism. For each $\sigma \geq 1$ there exists $H(\sigma) \geq 1$ with the following property. If there exists σ such that* $\text{skew}(f(T)) \leq \sigma$ *for all equilateral triangles $T \subset U$, then, for any $z \in U$ and any $r < \text{dist}(z, \mathbb{C} \setminus U)$, the following inequality $M(z,r) \leq Hm(z,r)$ holds where $H = H(\sigma)$. In particular, the map f is quasiconformal.*

Since quasiconformal maps are differentiable almost everywhere, we may improve the distortion bounds of quasiconformality. Let $\text{Skew}(f)$ denote the supremum of $\text{skew}(f(T))$ over all equilateral triangles contained in U; for $z \in U$ and $r > 0$, let $\text{skew}(f, z, r)$ denote the least upper bound of $\text{skew}(f(T))$ over all $T \subset B(z,r) \cap U$. Set $\text{skew}(f, z) = \liminf_{r \to 0} \text{skew}(f, z, r)$ and $\text{skew}(f) = ||\text{skew}(f, z)||_\infty$.

Corollary 3. *Let U be a domain in the complex plane $\mathbb{C}$, and let $f : U \to f(U)$ be an orientation-preserving homeomorphism with finite* $\text{Skew}(f)$*. If* $\text{skew}(f) \leq \sigma$ *then f is $K(\sigma)$-quasiconformal where*

$$K(\sigma) = \frac{\sqrt{\sigma^4 + 4\sigma^2 + 1 + 2\sigma\sqrt{3(\sigma^4 + \sigma^2 + 1)}} + \sigma^2 - 1}{\sqrt{\sigma^4 + 4\sigma^2 + 1 + 2\sigma\sqrt{3(\sigma^4 + \sigma^2 + 1)}} - (\sigma^2 - 1)}.$$

In particular, if $\text{skew}(f) = 1$ *then f is a conformal mapping.*

Most of the work in this chapter is my own. However, I would like to thank Aimo Hinkkanen and Peter Haïssinsky for beginning this project and bringing it to my attention. Many of their central ideas remain present in Lemma 7 and in the proof of Proposition 3. Hinkkanen proposed the method used in 5.5 and Haïssinsky suggested Corollary 3.

5.2 Proof of the Main Theorem

Throughout the rest of this chapter we will use the following notation and conventions:

(1) We define $\overline{B}(z,r) = \{w \in \mathbb{C} : |z-w| \leq r\}$ and let $C(z,r)$ be the boundary of $\overline{B}(z,r)$.

(2) By a curve we mean the image of a not necessarily one-to-one continuous function from a closed interval into $\mathbb{C}$.

(3) All triangles will be closed Euclidean triangles.

(4) Let $\mathcal{F}_\sigma$ denote the set of orientation-preserving homeomorphisms of any domain $U \subseteq \mathbb{C}$ into any domain $V \subseteq \mathbb{C}$ such that $\text{skew}(f(T)) \leq \sigma$ for all closed equilateral triangles $T \subset U$.

The proof of Theorem 6 relies on the following proposition.

Proposition 2. *Let U be a neighborhood of $\overline{B}(0,1)$, and let $f : U \to \mathbb{C}$ be a homeomorphism onto its image such that $f \in \mathcal{F}_\sigma$. Let T be the triangle with vertices 0, 1 and $\omega = 1/2 + (\sqrt{3}/2)i$. Then there exists a disk D contained in $f(T)$ such that*

(1) D is centered at $f(p)$ where $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i \approx 0.5 + 0.29i$, and

(2) there exists a constant $\alpha = \alpha(\sigma)$ such that the radius of D is at least $\alpha L(f(T))$.

We note that if f is to be quasiconformal, then, certainly, the image $f(T)$ has to contain a disk of definite size centered at the image of the centroid of the triangle, i.e., the point $\xi = 1/2 + (\sqrt{3}/6)i$. Unfortunately, its arithmetic properties make it difficult to relate this point to the vertices of T using equilateral triangles. The point p was chosen, because it is both close to the centroid ($|\xi - p| = \sqrt{3}/(2^9 \cdot 3)$), and it is a vertex of a tiling of the plane by equilateral triangles whose vertices include the vertices of T. Indeed, we have $p = 1/2 - 85 \cdot 2^{-9} + 85 \cdot 2^{-8}\omega$, cf. Lemma 7.

We first derive the proof of Theorem 6 from Proposition 2. We will then focus on the proof of the latter.

Proof of Theorem 6. Fix $z \in U$ and $r > 0$. If $\overline{B}(z,r) \subset U$, let $M(z,r) = \max\{|f(z) - f(w)| : w \in C(z,r)\}$ and $m(z,r) = \min\{|f(z) - f(w)| : w \in C(z,r)\}$. Denote by z_M a point in $C(z,r)$ such that $|f(z_M) - f(z)| = M(z,r)$.

Since $\mathcal{F}_\sigma$ is invariant under pre- and post-composition by affine maps, we may assume that $z = 0$, $r = 1$, and $z_M = 1$.

Let T_1 be the equilateral triangle with vertices $0, 1$ and ω. Then by Proposition 2, the image of T_1 must contain a disk centered at $f(p)$ and of radius at least $\alpha L(f(T_1))$.

Let us consider the isometry $A(z) = \overline{z - p}$. Let $T_2 = A(T_1)$. The triangle T_2 is contained in the unit disk, and A maps p to 0 and 1 to p. Since the other vertices of T_2 lie outside of T_1, we have $L(f(T_2)) \geq \alpha L(f(T_1))$. Moreover, another application of Proposition 2 implies that $f(T_2)$ contains the disk $\overline{B}(f(0), \alpha L(f(T_2)))$.

Summing up these estimates, we obtain

$$m(0,1) \geq \alpha L(f(T_2)) \geq \alpha^2 L(f(T_1)) \geq \alpha^2 M(0,1)\,.$$

$\square$

5.3 Construction of Certain Triangles

Proposition 2 is a consequence of the following proposition.

Proposition 3. *Let U be a neighborhood of $\overline{B}(0,1)$, and let $f : U \to \mathbb{C}$ be a homeomorphism onto its image such that $f \in \mathcal{F}_\sigma$. Let $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i$, $q = p + 2^{-9}$, and T be the closed triangle with vertices 0, 1 and ω. Then there exist points $t_1, t_2 \in T$ such that the points q, t_1, t_2 form the vertices of an equilateral triangle and the inequalities $|f(t_j) - f(p)| \leq C\mu$, and $|f(p) - f(q)| \geq cL(f(T))$ hold for some constants $c = c(\sigma)$ and $C = C(\sigma)$ where $\mu = \operatorname{dist}(f(p), \mathbb{C} \setminus f(T))$. We permit the trivial triangle where we have $t_1 = t_2 = q$.*

Proof of Proposition 2 assuming Proposition 3. If $t_1 = t_2 = q$, then we have

$$cL(f(T)) \leq |f(p) - f(q)| \leq C\mu.$$

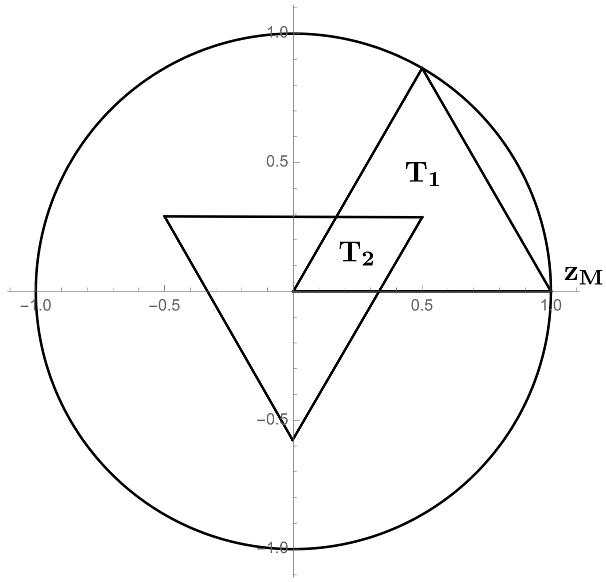

Figure 5.1: Configuration of $C(0,1)$, T_1 and T_2

Otherwise, by the triangle inequality

$$|f(p)-f(q)|-|f(t_1)-f(p)| \le |f(t_1)-f(q)| \le \sigma|f(t_1)-f(t_2)| \le \sigma(|f(t_1)-f(p)|+|f(p)-f(t_2)|)$$

so that by assumption,

$$cL(f(T))-C\mu \le 2\sigma C\mu \quad \text{whence} \quad \mu \ge \frac{c}{(2\sigma+1)C}L(f(T))\,.$$

□

5.4 Proof of Proposition 3

The idea of our proof of Proposition 3 is to define a curve γ' going through p such that

(1) for all $t \in \gamma'$ we have $|f(t)-f(p)| \le \sigma\mu(1+2\sigma^3)$;

(2) there are two points $t_1, t_2 \in \gamma'$, such that q, t_1, t_2 form the vertices of an equilateral triangle.

The proof of Proposition 3 results from Lemma 7 and Lemma 9.

We first prove the following result.

Lemma 7. *Suppose f satisfies the assumptions of Proposition 3. Let $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i$, $q = p + 2^{-9}$, and T be the closed triangle with vertices 0, 1 and ω. Then $|f(q) - f(p)| \geq cL(f(T))$ for some positive constant $c = c(\sigma)$.*

Proof. Let us first consider the tiling of the plane by equilateral triangles with vertices in $\Lambda = \mathbb{Z} \oplus \omega\mathbb{Z}$. Define a chain of triangles $(T_j)_{0 \leq j \leq J}$ as a sequence of triangles with vertices in Λ such that $T_j \cap T_{j+1}$ is an edge for all j with $0 \leq j < J$. Given two edges (v, w) and (v', w'), we may connect them by a chain of minimal length $n \geq 0$. A simple induction argument implies

$$|f(v) - f(w)| \leq \sigma^n |f(v') - f(w')|$$

if $f \in \mathcal{F}_\sigma$ is defined in a neighborhood of the chain.

Let T be as defined in our hypotheses: it is tiled by $N = 2^{18}$ triangles of $2^{-9}\Lambda$, and $[p, q]$ is an edge of this tiling. Therefore, for any other edge $[v, w]$, it follows that

$$|f(v) - f(w)| \leq \sigma^N |f(p) - f(q)|.$$

But each side of T is the union of less than N edges of our tiling, therefore, the triangle inequality implies

$$L(f(T)) \leq N\sigma^N |f(p) - f(q)|\,.$$

$\square$

We now prove a geometric lemma which will be used in the proof of Lemma 9.

Lemma 8. *Let $|z| \leq 1/8$ and suppose that $|\theta_\pm - (\pm\pi/3)| \leq 1/8$. Then the angle θ between $e^{i\theta_+} - z$ and $e^{i\theta_-} - z$ which crosses the positive real axis belongs to $(\pi/3, \pi)$.*

Proof. We note that $\cos\theta_\pm \geq 1/2 - 1/4 > 1/8 \geq |z|$ so that θ is less than π.

For the other inequality, we will estimate $\tan|\arg(e^{i\theta_\pm}-z)|$ to obtain a lower bound of both angles with the horizontal line:

$$\begin{aligned}\tan|\arg(e^{i\theta_\pm}-z)| &\geq \frac{\sqrt{3}/2-(|z|+1/8)}{1/2+(|z|+1/8)} \geq \frac{\sqrt{3}/2-1/4}{1/2+1/4}\\ &\geq \frac{2\sqrt{3}-1}{3} \geq \frac{2}{3} > \tan(\pi/6).\end{aligned}$$

Therefore θ is at least $\pi/3$. □

Now we demonstrate how to find the curve γ' mentioned above.

Lemma 9. *Under the assumptions of Proposition 3, there exists a curve γ' going through p such that for all $t \in \gamma'$ we have*

$$|f(t)-f(p)| \leq \sigma\mu(1+2\sigma^3)$$

and there are two points $t_1, t_2 \in \gamma'$, such that q, t_1, t_2 form the vertices of an equilateral triangle. We permit the trivial triangle where we have $t_1 = t_2 = q$.

Proof. We will do this in several steps. We first define a curve that will join two points of the boundary of a disk contained in T (Step 1). To make sure that we will be able to find two points that form an equilateral triangle with q, we will extend this curve so that it has end points on a slightly larger disk, and is only close to the boundary of the larger disk when it is also close to its end points (Step 2). Then we will use Lemma 8 to find our triangle (Step 3).

Since $\sqrt{3} \geq 8/5$, it follows that $\operatorname{dist}(p, \partial T) = 85\sqrt{3}\cdot 2^{-9} > 1/4 + 2^{-6}$, so that $\overline{B}(p, 1/4+2^{-6})$ is contained in the interior of T.

Throughout the proof, for $x \in \mathbb{C}$, R_x will denote the rotation centered at x of angle $\pi/3$, defined by $R_x(z) = x + (z-x)\omega$ and $\bar{R}_x$ the rotation centered at x of angle $-\pi/3$, defined by $\bar{R}_x(z) = x + (z-x)\bar{\omega}$. Recall that we set $\omega = 1/2 + (\sqrt{3}/2)i$.

Step 1: There exists a curve γ_2 that satisfies the following:

(1) $\gamma_2 \subset \overline{B}(p, 1/4)$**,**

(2) γ_2 **has end points on** $C(p, 1/4)$ **which are exactly** $2\pi/3$ **radians apart, and**

(3) for all points $t \in \gamma_2$ **we have** $|f(t) - f(p)| \leq \sigma\mu$**.**

Let $p' \in \partial T$ be such that $d(f(p), f(p')) = \mu$ and let $\gamma = f^{-1}([f(p), f(p')])$. Since $\overline{B}(p, 1/4)$ is contained in the interior of T, we may consider the component γ_1 of $\gamma \cap \overline{B}(p, 1/4)$ that contains p, and we denote by $w \in C(p, 1/4)$ the other end point of γ_1.

Now define

$$\gamma_2 = R_p(\gamma_1) \cup \bar{R}_p(\gamma_1)\,.$$

Note that, for any $s \in \gamma_1$, $R_p(s)$ and $\bar{R}_p(s)$ are two points in γ_2 which make an angle of $2\pi/3$ seen from p. Since $f \in \mathcal{F}_\sigma$, for all $t \in \gamma_2$, we have

$$|f(t) - f(p)| \leq \sigma |f(s) - f(p)| \leq \sigma\mu$$

where $s \in \gamma_1$ denotes a point such that either $t = R_p(s)$ or $t = \bar{R}_p(s)$.

Step 2: Let a, b **be the end points of** γ_2**. There exists a curve** γ_3 **such that**

(1) $\gamma_3 \subset \overline{B}(p, 1/4) \cup \overline{B}(a, 2^{-6}) \cup \overline{B}(b, 2^{-6})$**;**

(2) γ_3 **has both end points on** $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$**;**

(3) for all points $t \in \gamma_3$**,** $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$.

Let $\overline{B_a} = \overline{B}(a, 2^{-6})$ and $\overline{B_b} = \overline{B}(b, 2^{-6})$. Let γ_{2a} and γ_{2b} be the components of $\gamma_2 \cap \overline{B_a}$ and $\gamma_2 \cap \overline{B_b}$ that have end points at a and b respectively.

Clearly γ_{2a} also has an end point on the boundary of $\overline{B_a}$. Let a' denote an end point of γ_{2a} on the boundary of $\overline{B_a}$. Use the tangent line to $\overline{B}(p, 1/4)$ at a to divide $\overline{B_a}$ in half, and then divide each half into thirds. So we have divided $\overline{B_a}$ into closed sectors of $\pi/3$ radians with three such sectors lying entirely outside of $\overline{B}(p, 1/4)$. Let S_a denote the middle sector lying completely outside of $\overline{B}(p, 1/4)$. Then there exists $n \in \{2, 3\}$ such that when γ_{2a} is rotated $n\pi/3$ radians in an appropriate direction about a, the image of a' under the rotation will lie in S_a. Let the image of γ_{2a} under this rotation be denoted by γ_{3a}.

Now we will bound the quantity $|f(t) - f(p)|$ where $t \in \gamma_{3a}$. Fix $t \in \gamma_{3a}$. Let t_0 be the point on γ_{2a} whose image under the rotation is t. Without loss of generality we will assume this rotation was clockwise. Let t_i denote the image of t_0 under a clockwise rotation of $i\pi/3$ radians where $i = 1, \ldots, n$ ($t = t_n$). Then since a, t_{i-1}, t_i ($1 \leq i \leq n$) form an equilateral triangle, we have

$$|f(t_i) - f(a)| \leq \sigma |f(t_{i-1}) - f(a)|.$$

Since $a, t_0 \in \gamma_2$ we have

$$|f(a) - f(t_0)| \leq |f(a) - f(p)| + |f(p) - f(t_0)| \leq 2\sigma\mu$$

and

$$|f(a) - f(p)| \leq \sigma\mu.$$

Thus since n is at most 3 we have

$$|f(t) - f(p)| \leq |f(a) - f(p)| + |f(a) - f(t)| \leq \sigma\mu + \sigma^n |f(a) - f(t_0)| \leq \sigma\mu(1 + 2\sigma^n) \leq \sigma\mu(1 + 2\sigma^3).$$

Furthermore γ_{3a} must intersect the circle $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$. This is because γ_{3a} has an end point in S_a and therefore the distance of the end point of γ_{3a} from $\overline{B}(p, 1/4)$ must be at least $\cos(\pi/6) \cdot 2^{-6} = \sqrt{3} \cdot 2^{-7}$. This is depicted in figure 5.2.

We proceed similarly near b and define a curve γ_{3b} contained in $\overline{B_b}$ with end points at b and at some point on the intersection of the boundary of $\overline{B_b}$ and S_b (defined analogously to S_a) such that for all $t \in \gamma_{3b}$ we have $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$; as above, γ_{3b} intersects the circle $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$.

Let γ_3 be the connected component of $(\gamma_2 \cup \gamma_{3a} \cup \gamma_{3b}) \cap \overline{B}(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ which includes points in both γ_{3a} and γ_{3b}. Then for all points $t \in \gamma_3$,

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3).$$

The curve γ' in Lemma 9 can be chosen as $\gamma' = \gamma_3$.

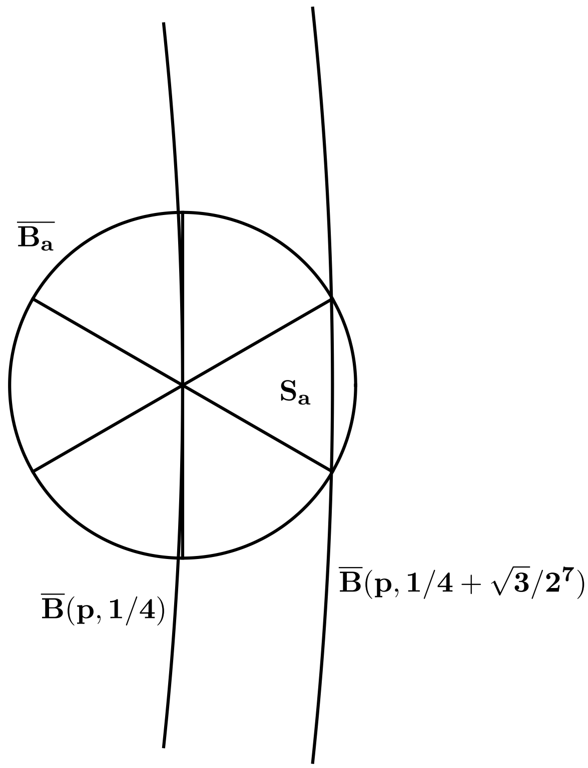

Figure 5.2:

Step 3: Let $q = p + 2^{-9}$. There exist $t_1, t_2 \in \gamma_3$ such that $\{q, t_1, t_2\}$ form an equilateral triangle.

Let $\overline{B_q}$ be the smallest disk centered at q which contains $\overline{B}(p, 1/4)$. Then $\overline{B_q} \subset \overline{B}(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ since $|p - q| = 2^{-9} \leq \sqrt{3} \cdot 2^{-8}$. Let γ_4 be the connected component of $\gamma_3 \cap \overline{B_q}$ which has end points $A \in \overline{B_a} \cap \overline{B_q}$ and $B \in \overline{B_b} \cap \overline{B_q}$.

Note that, if we write $a = p + |a-p|e^{i\theta_a}$ and $A = p + |A-p|e^{i\theta_A}$, then

$$|\theta_A - \theta_a| \leq 2|a-A|/(1/4) \leq 2 \cdot 2^{-6}/(1/4) \leq 1/8$$

and similarly for b and B. Note that $|\arg(a-p)|+|\arg(b-p)| = 2\pi/3$ and $|p-q|/(1/4) = 2^{-11} \leq 1/8$. Therefore, by Lemma 8 applied in $\overline{B}(p, 1/4)$, the angle between $A-q$ and $B-q$ lies in $(\pi/3, \pi)$. Hence, the images A_r and B_r of A and B respectively under $\bar{R}_q$ will separate A and B on $\partial\overline{B_q}$. Thus the image $\bar{R}_q(\gamma_4)$ must intersect γ_4. This gives us our desired equilateral triangle since q, the intersection point, and the pre-image of the intersection point form an equilateral triangle.

This completes the proof of Lemma 9. □

5.5 An Alternative Proof of Theorem 2 from Proposition 2

This proof parallels Pfluger's proof that a mapping satisfying the geometric definition of quasi-conformality is absolutely continuous on lines. His proof can be found in [21] and is reproduced in English in [13], p. 162. Throughout our proof we make use of Proposition 2 by noting that through conjugation by a Möbius transformation, this proposition tells us that the image of every equilateral triangle, T, contains a disk with radius proportional to $L(f(T))$.

Lemma 10. *Let f be as in Proposition 2. Then f is absolutely continuous on lines.*

Proof. Fix a horizontal rectangle $R = \{(x,y) : a < x < b, c < y < d\}$. Let $I_y = \{(x,y) : a < x < b\}$ for y between c and d and define $A(y)$ to be the area in $f(R)$ beneath the image of the line segment I_y. Since A is an increasing function, A is differentiable for almost every y between c and d. Fix such a y where $A'(y)$ exists, and let $\{(z_k^*, z_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of I_y where $z_k = (x_k, y)$ and $z_k^* = (x_k^*, y)$.

Now fix a k and draw a rectangle of height δ above (z_k^*, z_k). Set N_k equal to the smallest integer greater than or equal to $\frac{|x_k^* - x_k|}{\delta}$. We will now draw N_k equilateral triangles in our rectangle. The triangles will all have one side on the interval $[z_k^*, z_k]$ and they will overlap only at their vertices. The first $N-1$ triangles will have width δ and the last triangle will have width $x_k^* - x_k - \delta(N-1)$. Let

Δ_{k_i} denote the side length of the ith triangle for $1 \le i \le N_k$ and set $\Delta_{k_0} = 0$. Set $N = \sum_{k=1}^{n} N_k$. Then $N \le \frac{\sum_{k=1}^{n} |x_k^* - x_k|}{\delta} + n$.

By Proposition 2, the image of each of our triangles must contain a disk of radius comparable to the greatest distance between the images of the vertices of the triangle. Thus the total area of all the images of our rectangles of height δ and width $x_k^* - x_k$ is greater than or equal to

$$\sum_{k=1}^{n} \sum_{j=1}^{N_k} \pi \left(\frac{|f((x_k + \sum_{i=1}^{j} \Delta_{k_i}, y)) - f((x_k + \sum_{i=1}^{j} \Delta_{k_{i-1}}, y))|}{\alpha} \right)^2 . \tag{5.5.1}$$

Then by Schwarz's inequality, (5.5.1) is greater than or equal to

$$\frac{\pi}{N\alpha^2} \left(\sum_{k=1}^{n} \sum_{j=1}^{N_k} |f((x_k + \sum_{i=1}^{j} \Delta_{k_i}, y)) - f((x_k + \sum_{i=1}^{j} \Delta_{k_{i-1}}, y))| \right)^2 \ge \frac{\pi}{N\alpha^2} \left(\sum_{k=1}^{n} |f(z_k^*) - f(z_k)| \right)^2$$

$$\ge \frac{\pi}{\left(\frac{\sum_{k=1}^{n} |x_k^* - x_k|}{\delta} + n \right) \alpha^2} \left(\sum_{k=1}^{n} |f(z_k^*) - f(z_k)| \right)^2$$

$$= \frac{\pi}{\left(\frac{\sum_{k=1}^{n} |x_k^* - x_k| + \delta n}{\delta} \right) \alpha^2} \left(\sum_{k=1}^{n} |f(z_k^*) - f(z_k)| \right)^2 .$$

Recall that $A(y)$ is defined to be the area in $f(R)$ beneath the image of the line segment I_y. Since the areas of the images of our rectangles of height δ and width $x_k^* - x_k$ are less than or equal to $A(y + \delta) - A(y)$, we have

$$\frac{\pi}{\alpha^2} \left(\sum_{k=1}^{n} |f(z_k^*) - f(z_k)| \right)^2 \le \left(\frac{A(y+\delta) - A(y)}{\delta} \right) \left(\sum_{k=1}^{n} |x_k^* - x_k| + \delta n \right) .$$

Since we chose y where A is differentiable, letting δ go to 0 gives

$$\frac{\pi}{\alpha^2} \left(\sum_{k=1}^{n} |f(z_k^*) - f(z_k)| \right)^2 \le A'(y) \left(\sum_{k=1}^{n} |x_k^* - x_k| \right) .$$

Since $A'(y)$ exists almost everywhere this gives absolute continuity on almost every horizontal line segment. The proof is analogous for vertical line segments. $\square$

Proof of Theorem 2. Since f is a homeomorphism and is absolutely continuous on lines, the partial derivatives of f exist almost everywhere, and hence f is differentiable almost everywhere [13], pp. 128-130. We will now show

$$\max_{\xi} |\partial_\xi f(z)| \leq K \min_{\xi} |\partial_\xi f(z)| \text{ a.e.} \tag{5.5.2}$$

Consider a point z where f is differentiable. Using a series of translations and rotations we can assume $z = 0$, $f(z) = 0$, $f_z(0) = |f_z(0)|$ and $f_{\bar{z}}(0) = |f_{\bar{z}}(0)|$. Let S_δ be the open square with side lengths 2δ centered at the origin and let S'_δ denote its image under f. Then

$$s_b(S'_\delta) = 2\delta(|f_z(0)| + |f_{\bar{z}}(0)|) + o(\delta)$$

and the area of S'_δ is

$$4\delta^2(|f_z(0)|^2 - |f_{\bar{z}}(0)|^2) + o(\delta^2). \tag{5.5.3}$$

Now we seek an upper bound on $\frac{s_b(S'_\delta)^2}{m(S'_\delta)}$ where $m(S'_\delta)$ denotes the area of S'_δ. This will replace the use of Rengel's inequality in the proof that the geometric definition implies the analytic definition. For our purposes, Rengel's inequality is not useful since we have no bound on the modulus of S'_δ. For a statement of Rengel's inequality see 3.1.

Let γ be a curve in $\overline{S_\delta}$ such that its image has length $s_b(S'_\delta)$. Denote the left end point of γ by z_1 and the right end point of γ by z_2. Let T_1 be the equilateral triangle contained in S_δ with one vertex at z_1 and one vertex at $(-\delta, 0)$. If $z_1 = (-\delta, 0)$ take T_1 to have all vertices equal to z_1, and for the subsequent discussion define $L(f(T_1)) = 0$ and $m(f(T_1)) = 0$. Let T_2 be the triangle with one vertex at $(-\delta, 0)$, another vertex at $(0, 0)$ and such that T_2 does not lie in the same quadrant as T_1. Let T_4 be the equilateral triangle contained in S_δ with one vertex at z_2 and one vertex at $(\delta, 0)$. Let cases where $z_2 = (\delta, 0)$ be handled in the same way as for T_1. Let T_3 be the triangle with one vertex at $(\delta, 0)$, another vertex at $(0, 0)$ and such that T_3 does not lie in the same quadrant as T_4. See figure 5.3 for an example of where to place the triangles T_1, T_2, T_3 and T_4 for a given choice of γ.

Note the interiors of $\{T_i\}_{i=1}^4$ are pairwise disjoint and each T_i shares at least one vertex with T_{i+1}.

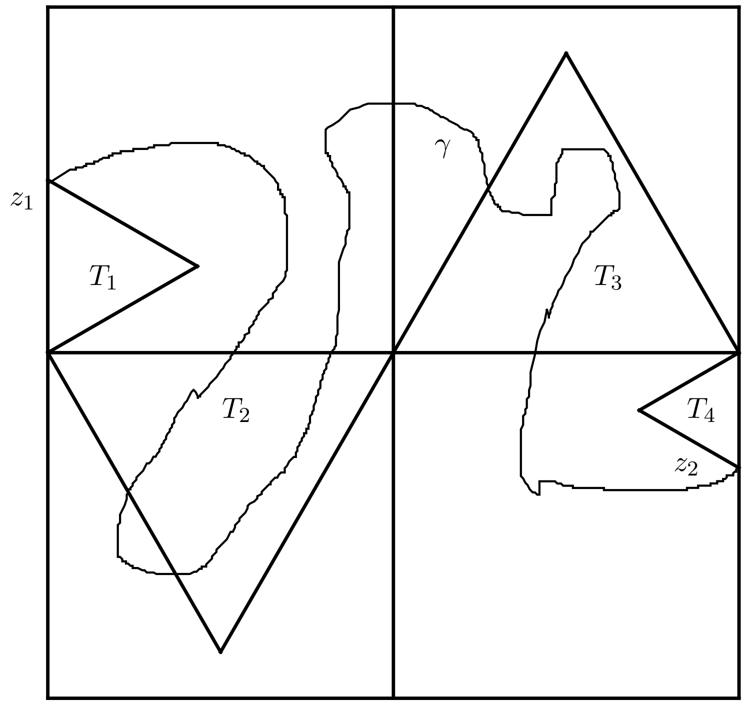

Figure 5.3: S_δ with the configuration of T_1, T_2, T_3 and T_4 for the depicted curve γ

Thus we have the following two inequalities:

$$s_b(S'_\delta) \leq |f(z_1) - f((-\delta,0))| + |f((-\delta,0)) - f((0,0))| + |f((0,0)) - f((\delta,0))| + |f((\delta,0)) - f(z_2)| \leq \sum_{i=1}^{4} L(f(T_i))$$

and

$$m(S'_\delta) \geq \sum_{i=1}^{4} m(f(T_i)) \geq \sum_{i=1}^{4} \pi \left(\alpha L(f(T_i))\right)^2$$

where $m(f(T_i))$ denotes the area of $f(T_i)$.

Now we have

$$\frac{s_b(S'_\delta)^2}{m(S'_\delta)} \leq \frac{\left(\sum_{i=1}^{4} L(f(T_i))\right)^2}{\sum_{i=1}^{4} \pi \left(\alpha L(f(T_i))\right)^2} \leq \frac{\left(\sum_{i=1}^{4} L(f(T_i))\right)^2}{\frac{1}{4}\left(\sum_{i=1}^{4} \pi\alpha L(f(T_i))\right)^2} = \frac{4}{(\pi\alpha)^2} \tag{5.5.4}$$

Thus combining with our previous formulas for $s_b(R'_\delta)$ and $m(R'_\delta)$, we have

$$\frac{4\delta^2(|f_z(0)|+|f_{\bar z}(0)|)^2+o(\delta^2)}{4\delta^2(|f_z(0)|^2-|f_{\bar z}(0)|^2)+o(\delta^2)}=\frac{s_b(S'_\delta)^2}{m(S'_\delta)}\leq\frac{4}{(\pi\alpha)^2} \tag{5.5.5}$$

Furthermore

$$\begin{aligned}
\lim_{\delta\to 0}\frac{4\delta^2(|f_z(0)|+|f_{\bar z}(0)|)^2+o(\delta^2)}{4\delta^2(|f_z(0)|^2-|f_{\bar z}(0)|^2)+o(\delta^2)} &= \lim_{\delta\to 0}\frac{(|f_z(0)|+|f_{\bar z}(0)|)^2+\frac{o(\delta^2)}{\delta^2}}{(|f_z(0)|^2-|f_{\bar z}(0)|^2)+\frac{o(\delta^2)}{\delta^2}}\\
&= \frac{(|f_z(0)|+|f_{\bar z}(0)|)^2}{|f_z(0)|^2-|f_{\bar z}(0)|^2}\\
&= \frac{|f_z(0)|+|f_{\bar z}(0)|}{|f_z(0)|-|f_{\bar z}(0)|}\\
&= \frac{\max_\xi|\partial_\xi f(z)|}{\min_\xi|\partial_\xi f(z)|}
\end{aligned}$$

Combining this with the previous inequality gives our desired result. □

5.6 Proof of Corollary 3

We prove Corollary 3 by approximating f by linear mappings at points where f is differentiable.

Proof. Theorem 6 implies that f is quasiconformal and hence differentiable at almost every point. Let z_0 be a point of differentiability such that $\mathrm{skew}(f,z_0)\leq\sigma$. We will compute the maximum possible value for $H(z_0)$. Since $H(z_0)$ is invariant under Möbius transformations we may compose with translations, a dilation and a rotation to assume that $z_0=f(z_0)=0$, $f_z(z_0)=1$ and $f_{\bar z}(z_0)=|f_{\bar z}(z_0)|<1$. Then $f(z)=z+f_{\bar z}(z_0)\bar z+\epsilon(z)$ where $\epsilon(z)/|z|$ tends to 0 as z tends to z_0, and thus $\mathrm{skew}(f,z_0)=\mathrm{skew}(\tilde f,z_0)$ where $\tilde f(z)=z+f_{\bar z}(z_0)\bar z$.

Note $|\tilde f(a)-\tilde f(b)|=|\tilde f(a+v)-\tilde f(b+v)|$, $|\tilde f(a)-\tilde f(b)|=|\tilde f(\bar a)-\tilde f(\bar b)|$ and $|\tilde f(a)-\tilde f(b)|/|\tilde f(a)-\tilde f(c)|=|\tilde f(ra)-\tilde f(rb)|/|\tilde f(ra)-\tilde f(rc)|$ for all $a,b,c,v\in\mathbb{C}$ with $a\neq c$ and all $r>0$. This implies $\mathrm{skew}(\tilde f(T))$ where T is an equilateral triangle is invariant under translations, conjugation and dilations of T. Thus for all equilateral triangles T,

$$\mathrm{skew}(\tilde{f}(T)) \in \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\}.$$

Indeed, suppose T has vertices A, B and C, and $\mathrm{skew}(T) = \frac{|\tilde{f}(A)-\tilde{f}(B)|}{|\tilde{f}(A)-\tilde{f}(C)|}$. First we translate A to the origin, and then we dilate T so its side lengths are equal to 1. If $\overline{AB}$ is $\pi/3$ radians clockwise from $\overline{AC}$, it is clear that our statement is true, Otherwise we take the complex conjugate of T to change the orientation of T and then, since $\tilde{f}$ is invariant under conjugations of T, our claim is true.

Hence we have

$$\mathrm{skew}(\tilde{f}) = \max \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\} = \max \left\{ \frac{|\tilde{f}(z)|}{|\tilde{f}(ze^{i\pi/3})|} : |z| = 1 \right\}.$$

Let $\mu = f_{\bar{z}}$, $\nu = 1 + \mu^2$, $\alpha = e^{i\pi/3}$ and $x = 2\arg(z)$. We have

$$|\tilde{f}(z)|^2 = |z + \mu\bar{z}|^2 = |z(1 + \mu\bar{z}^2)|^2 = |1 + \mu\bar{z}^2|^2 = 1 + \mu^2 + 2\mu\mathrm{Re}(z^2) = \nu + 2\mu\cos(x).$$

Replacing z with $z\alpha$ in the above calculation gives

$$|\tilde{f}(z\alpha)|^2 = \nu + 2\mu\mathrm{Re}(z^2\alpha^2) = \nu + 2\mu\cos(x + 2\pi/3) = \nu - \mu\cos(x) - \sqrt{3}\mu\sin(x).$$

Now we are able to maximize $|\tilde{f}(z)|/|\tilde{f}(\alpha z)|$ with respect to z. Since we have assumed $|z| = 1$, we can instead maximize $\kappa(x) = |\tilde{f}(e^{ix/2})|^2/|\tilde{f}(\alpha e^{ix/2})|^2$ with respect to x and then take the square root. Differentiating $\kappa(x)$ we obtain

$$\kappa'(x) = \frac{(\nu - \mu\cos(x) - \sqrt{3}\mu\sin(x))(-2\mu\sin(x)) - (\nu + 2\mu\cos(x))(\mu\sin(x) - \sqrt{3}\mu\cos(x))}{(\nu - \mu\cos(x) - \sqrt{3}\mu\sin(x))^2}$$

$$= \frac{2\sqrt{3}\mu^2 + \sqrt{3}\mu\nu\cos(x) - 3\mu\nu\sin(x)}{(\nu - \mu\cos(x) - \sqrt{3}\mu\sin(x))^2}.$$

Setting $\kappa'(x)$ equal to zero gives

$$2\sqrt{3}\mu^2 = 3\mu\nu\sin(x) - \sqrt{3}\mu\nu\cos(x)$$

from which it follows that

$$\mu/\nu = \sin(2\pi/3)\sin(x) + \cos(2\pi/3)\cos(x)$$

and thus

$$\mu/\nu = \cos(x - 2\pi/3).$$

Therefore our maximum must be attained at $x = 2\pi/3 \pm \arccos(\mu/\nu)$.

Set $z_j = e^{ix_j/2}$ where $x_1 = 2\pi/3 + \arccos(\mu/\nu)$ and $x_2 = 2\pi/3 - \arccos(\mu/\nu)$. Then

$$|\tilde{f}(z_1)|^2 = \nu + 2\mu\cos\left(\frac{2\pi}{3} + \arccos\left(\frac{\mu}{\nu}\right)\right) = \nu + 2\mu\left(\frac{-\mu}{2\nu} - \frac{\sqrt{3}}{2}\frac{\sqrt{\nu^2-\mu^2}}{\nu}\right) = \nu - \frac{\mu^2}{\nu} - \sqrt{3}\mu\frac{\sqrt{\nu^2-\mu^2}}{\nu},$$

$$|\tilde{f}(z_2)|^2 = \nu + 2\mu\cos\left(\frac{2\pi}{3} - \arccos\left(\frac{\mu}{\nu}\right)\right) = \nu + 2\mu\left(\frac{-\mu}{2\nu} + \frac{\sqrt{3}}{2}\frac{\sqrt{\nu^2-\mu^2}}{\nu}\right) = \nu - \frac{\mu^2}{\nu} + \sqrt{3}\mu\frac{\sqrt{\nu^2-\mu^2}}{\nu},$$

$$\begin{aligned}
|\tilde{f}(z_1\alpha)|^2 &= \nu - \mu\cos\left(\frac{2\pi}{3} + \arccos\left(\frac{\mu}{\nu}\right)\right) - \sqrt{3}\mu\sin\left(\frac{2\pi}{3} + \arccos\left(\frac{\mu}{\nu}\right)\right) \\
&= \nu + \frac{\mu^2}{2\nu} + \frac{\sqrt{3}\mu}{2}\frac{\sqrt{\nu^2-\mu^2}}{\nu} - \frac{3\mu^2}{2\nu} + \frac{\sqrt{3}\mu}{2}\frac{\sqrt{\nu^2-\mu^2}}{\nu} \\
&= \nu - \frac{\mu^2}{\nu} + \sqrt{3}\mu\frac{\sqrt{\nu^2-\mu^2}}{\nu},
\end{aligned}$$

and

$$
\begin{aligned}
|\tilde{f}(z_2\alpha)|^2 &= \nu - \mu\cos\left(\frac{2\pi}{3} - \arccos\left(\frac{\mu}{\nu}\right)\right) - \sqrt{3}\mu\sin\left(\frac{2\pi}{3} - \arccos\left(\frac{\mu}{\nu}\right)\right) \\
&= \nu + \frac{\mu^2}{2\nu} - \frac{\sqrt{3}\mu}{2}\frac{\sqrt{\nu^2-\mu^2}}{\nu} - \frac{3\mu^2}{2\nu} - \frac{\sqrt{3}\mu}{2}\frac{\sqrt{\nu^2-\mu^2}}{\nu} \\
&= \nu - \frac{\mu^2}{\nu} - \sqrt{3}\mu\frac{\sqrt{\nu^2-\mu^2}}{\nu}.
\end{aligned}
$$

Thus

$$
\operatorname{skew}(\tilde{f}) = \sqrt{\frac{\nu - \frac{\mu^2}{\nu} + \sqrt{3}\mu\frac{\sqrt{\nu^2-\mu^2}}{\nu}}{\nu - \frac{\mu^2}{\nu} - \sqrt{3}\mu\frac{\sqrt{\nu^2-\mu^2}}{\nu}}} = \sqrt{\frac{\sqrt{\nu^2-\mu^2}+\sqrt{3}\mu}{\sqrt{\nu^2-\mu^2}-\sqrt{3}\mu}}. \tag{5.6.1}
$$

We set $\operatorname{skew}(\tilde{f}) = \sigma$ and our goal is to solve for μ. Set $a = \frac{\sqrt{3}\mu}{\sqrt{\nu^2-\mu^2}}$. Then

$$
\sigma^2 = \frac{1+a}{1-a}
$$

and

$$
a = \frac{\sigma^2-1}{\sigma^2+1}.
$$

We will now solve for μ in terms of a and then substitute $\frac{\sigma^2-1}{\sigma^2+1}$ for a. We get

$$
\begin{aligned}
a^2(\nu^2-\mu^2) = 3\mu^2 \quad &\Longrightarrow \quad a^2((\mu^2+1)^2-\mu^2) = 3\mu^2 \\
&\Longrightarrow \quad a^2(\mu^4+\mu^2+1) = 3\mu^2 \\
&\Longrightarrow \quad a^2\mu^4 + (a^2-3)\mu^2 + a^2 = 0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mu^2 &= \frac{3-a^2-\sqrt{(a^2-3)^2-4a^4}}{2a^2} \\
&= \frac{3-\left(\frac{\sigma^2-1}{\sigma^2+1}\right)^2 - \sqrt{\left(\left(\frac{\sigma^2-1}{\sigma^2+1}\right)^2-3\right)^2 - 4\left(\frac{\sigma^2-1}{\sigma^2+1}\right)^4}}{2\left(\frac{\sigma^2-1}{\sigma^2+1}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3(\sigma^2+1)^2-(\sigma^2-1)^2-\sqrt{((\sigma^2-1)^2-3(\sigma^2+1)^2)^2-4(\sigma^2-1)^4}}{2(\sigma^2-1)^2} \\
&= \frac{2\sigma^4+8\sigma^2+2-\sqrt{48\sigma^2(\sigma^4+\sigma^2+1)}}{2(\sigma^2-1)^2} \\
&= \frac{\sigma^4+4\sigma^2+1-2\sigma\sqrt{3(\sigma^4+\sigma^2+1)}}{(\sigma^2-1)^2}
\end{aligned}
$$

and finally we have

$$
\begin{aligned}
\mu &= \frac{\sqrt{\sigma^4+4\sigma^2+1-2\sigma\sqrt{3(\sigma^4+\sigma^2+1)}}}{\sigma^2-1} \\
&= \frac{\sqrt{\frac{(\sigma^4+4\sigma^2+1)^2-12\sigma^2(\sigma^4+\sigma^2+1)}{\sigma^4+4\sigma^2+1+2\sigma\sqrt{3(\sigma^4+\sigma^2+1)}}}}{\sigma^2-1} \\
&= \frac{\sqrt{\frac{(\sigma^2-1)^4}{\sigma^4+4\sigma^2+1+2\sigma\sqrt{3(\sigma^4+\sigma^2+1)}}}}{\sigma^2-1} \\
&= \frac{\sigma^2-1}{\sqrt{\sigma^4+4\sigma^2+1+2\sigma\sqrt{3(\sigma^4+\sigma^2+1)}}}.
\end{aligned}
$$

Note that when using the quadratic formula in our calculations we choose the negative square root, because this choice gives values of μ in $[0, 1)$. Hence

$$
K(\sigma) = \frac{\sqrt{\sigma^4+4\sigma^2+1+2\sigma\sqrt{3(\sigma^4+\sigma^2+1)}}+\sigma^2-1}{\sqrt{\sigma^4+4\sigma^2+1+2\sigma\sqrt{3(\sigma^4+\sigma^2+1)}}-(\sigma^2-1)}.
$$

□

5.7 An Analogue of the Main Theorem in Hilbert Spaces of Dimension at Least Three

In dimensions three and higher the proof of an analogue of Theorem 6 is surprisingly simpler than the proof of Theorem 6. Furthermore the proof itself gives an elegant bound on $K(\sigma)$.

Theorem 7. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional Hilbert spaces with $dim(\mathcal{H}_1) = dim(\mathcal{H}_2) \geq 3$ and* *let $U \subset \mathcal{H}_1, V \subset \mathcal{H}_2$ be domains. Suppose $f : U \to V$ is a homeomorphism and that for all closed equilateral triangles $T \subset U$, $skew(f(T)) \leq \sigma$. Then f is σ^3-quasiconformal when using the metric definition of quasiconformality.*

Proof. It suffices to assume $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^n$ for some $n \geq 3$. We will show that f satisfies the metric definition of quasiconformality.

Fix a point $p \in U$, a positive number r with $r < \operatorname{dist}(p, \partial U)$ and $a \in \partial B(p,r)$. Let $m \in \partial B(p,r)$ be such that

$$|f(p) - f(m)| = \min_{r=|z-p|} |f(z) - f(p)|.$$

We will prove $|f(a) - f(p)| \leq \sigma^3 |f(p) - f(m)|$. Let $\mathbf{e_i}$ denote the unit vector in the ith direction. To simplify our calculations we will actually show $|f(a') - f(p')| \leq K^3 |f(p') - f(m')|$ where a', p' and m' are the images of the points a, p and m respectively under a sequence of conformal mappings, and f is modified accordingly without changing notation. Namely, first apply a translation so that $p' = 0$, then a rotation so that $m' = r\mathbf{e_1}$ and $a' = a_1\mathbf{e_1} + a_2\mathbf{e_2}$ and finally a possible reflection so that $a_2 > 0$. From now on we will only work in the linear subspace spanned by the first three coordinates which we will identify with $\mathbb{R}^3$. More precisely, we will identify $\mathbf{e_1}$ with the unit vector in the x direction, $\mathbf{e_2}$ with the unit vector in the y direction and another arbitrary coordinate with the z direction.

Case 1: If the smaller angle between a' and the positive x-axis is less than or equal to $2\pi/3$ then set

$$b = \left(r/2, r(1-a_1)/(2a_2), r\sqrt{3/4 - [(1-a_1)/(2a_2)]^2}\right).$$

Consider the triangles T_1 with vertices p', m' and b, and T_2 with vertices p', b and a'. The triangles T_1 and T_2 are equilateral triangles which share a common side with endpoints at b and p'.

Thus

$$|f(p') - f(a')| \leq \sigma |f(p') - f(b)| \leq \sigma^2 |f(p') - f(m')|$$

Case 2: If the smaller angle between a' and the x-axis is not less than or equal to $2\pi/3$, consider the equilateral triangle T_0 with vertices p', a' and b' where b' is the image of a' under a rotation of $\pi/3$ radians clockwise. The smaller angle between b' and the x-axis is less than or equal to $2\pi/3$.

Thus by Case 1

$$|f(p') - f(b')| \leq \sigma^2 |f(p') - f(m')|.$$

Then since the triangle T_0 has sides with endpoints at p' and a', and p' and b' we have

$$|f(p') - f(a')| \leq \sigma |f(p') - f(b')| \leq \sigma^3 |f(p') - f(m')|.$$

□

Chapter 6

Quasiconformal Mappings on the Grushin Plane

6.1 Introduction

In this chapter we will study quasiconformal mappings on a generalization of the Grushin plane referred to as the r-Grushin plane. To the best of our knowledge the following definition is original.

Definition 8. *Let $r : \mathbb{R} \to \mathbb{R}$ be a differentiable homeomorphism satisfying the following properties:*
1. r' is an even function and $r'|_{[0,\infty)}$ is a homeomorphism of $[0, \infty)$ onto itself.
2. There exists $\beta > 1$ such that for all $u \in \mathbb{R} \setminus \{0\}$

$$\frac{r(u)}{u} \leq r'(u) \leq \beta \frac{r(u)}{u}.$$

The r-Grushin plane G_r is $\mathbb{R}^2$ with the metric defined by the Carnot-Carathéodory distance

$$d_{rCC}(w, w') = \inf \ell_r(\gamma)$$

where the infimum is taken over all absolutely continuous, horizontal paths $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to G_r$ with $\gamma(0) = w$ and $\gamma(1) = w'$, and the length of γ is defined by

$$\ell_r(\gamma) = \ell_r(\gamma_1, \gamma_2) = \int_0^1 \sqrt{(\gamma_1'(s))^2 + \frac{(\gamma_2'(s))^2}{(r'(\gamma_1(s)))^2}} ds.$$

Just as on the classical Grushin plane, paths on G_r are called horizontal if they have a horizontal tangent at every point where they cross the vertical axis. Throughout this chapter we take (u, v) to be the coordinates on G_r.

The simplest example of homeomorphisms r satisfying Definition 8 are the power functions used by Meyerson which were mentioned in 3.4.2. Another slightly more complex class of examples are functions of the form

$$r(u) = \begin{cases} u^p \ln(u+1) & u \geq 0 \\ -|u|^p \ln(|u|+1) & u < 0 \end{cases}$$

where $p > 1$. The reader can easily check that these satisfy the requirements of our definition. In this chapter we determine conditions on a homeomorphism $r : \mathbb{R} \to \mathbb{R}$ such that $\Phi_r : (u, v) \to r(u) + iv$ is a quasisymmetry between the complex plane and the metric space G_r defined by the vector fields $\frac{\partial}{\partial u}$ and $r'(u)\frac{\partial}{\partial v}$. These quasisymmetries are useful to us, because they can be used to translate the rich theory of quasiconformal mappings in the complex plane to the G_r spaces via conjugation. For example, we can define the r-Grushin Beltrami equation as follows:

Definition 9. *Suppose $g = (g_1, g_2) : G_r \to G_r$ and define $\tilde{g} = \Phi_r \circ g$, $W = \frac{1}{2}(\frac{\partial}{\partial u} - ir'(u)\frac{\partial}{\partial v})$ and $\overline{W} = \frac{1}{2}(\frac{\partial}{\partial u} + ir'(u)\frac{\partial}{\partial v})$. We say $\tilde{g}$ satisfies the r-Grushin Beltrami equation provided that*

$$\overline{W}\tilde{g} = \nu W \tilde{g} \ \ a.e. \tag{6.1.1}$$

where ν is some measurable function with $||\nu||_\infty < 1$.

Then we obtain an analytic characterization of quasisymmetry in G_r.

Theorem 8. *A map $g : G_r \to G_r$ is quasisymmetric if and only if $\tilde{g}$ is a homeomorphism that is absolutely continuous on lines, and satisfies equation (6.1.1) for all points at which it is defined.*

In 6.5 we will seek to reconcile this theorem with notions of conformal mappings. For example, we will generalize the definition of conformality on Riemannian manifolds to develop the following definition:

Definition 10. *Suppose A and B are domains in G and $g = (g_1, g_2) : A \to B$ is a homeomorphism. Define $A' = A \setminus \{u = 0 \text{ or } g_1(u) = 0\}$. We say $g|_{A'}$ is conformal provided that*

$$Dg = \begin{pmatrix} \frac{\partial g_1}{\partial u} & |u|\frac{\partial g_1}{\partial v} \\ \frac{1}{|g_1|}\frac{\partial g_2}{\partial u} & \frac{|u|}{|g_1|}\frac{\partial g_2}{\partial v} \end{pmatrix}$$

is defined and is a conformal matrix for every point in A'. We say g is conformal on all of A if g is conformal on A' and for all points $w_0 \in A - A'$,

$$\lim_{w \to w_0} D_r g(w)$$

is defined and non-zero. We take the limit along all paths in A'.

We will show with certain conditions this definition is equivalent to g being quasisymmetric and ν being identically zero. Furthermore, our definition is satisfied by a class of conformal maps on the Grushin plane discovered by Payne [20].

6.2 A Quasidistance for the G_r Spaces

Since the Carnot-Carathéodory distance does not lend itself to proving quasisymmetry directly we will define a quasidistance d_r, and then show it suffices to only consider the quasidistance. More precisely, we will show there exists a constant C such that if $w, a, b \in G_r$ and $d_{rCC}(w, a) \leq d_{rCC}(w, b)$, then $d_r(w, a) \leq C d_r(w, b)$.

The following lemma gives another property of r and will be used throughout our proof of quasisymmetry.

Lemma 11. *As defined above the function r' is doubling when restricted to $[0, \infty)$. In other words, there exists a constant $m > 0$ such that for all $u \in [0, \infty)$ we have $r'(2u) \leq m r'(u)$.*

Proof. First we show $r|_{(0,\infty)}$ is doubling. Choose $\alpha > 1$ such that $\beta \ln \alpha < 1$. By our conditions on r we have $r(\alpha u) = \int_u^{\alpha u} r'(t)dt + r(u) \leq \int_u^{\alpha u} \beta \frac{r(t)}{t} dt + r(u) \leq \beta r(\alpha u) \int_u^{\alpha u} \frac{dt}{t} + r(u) = \beta r(\alpha u) \ln \alpha + r(u)$. Thus $r(\alpha u) \leq \frac{r(u)}{1 - \beta \ln \alpha}$ where $\frac{1}{1 - \beta \ln \alpha} > 0$. Since $\alpha > 1$ and $r|_{(0,\infty)}$ is increasing, repeated iteration gives $r|_{(0,\infty)}$ is doubling for some constant m. Then since

$$\frac{2u}{\beta} r'(2u) \leq r(2u) \leq m r(u) \leq m u r'(u),$$

r' restricted to $(0, \infty)$ is also doubling. The claim is trivial for $u = 0$. $\square$

The definition below is a generalization of Meyerson's quasidistance [16].

Definition 11. *The r-Grushin quasidistance between two points $w, w' \in G_r$ is*

$$d_r(w, w') = \max\left\{|u - u'|, \min\left\{M, \frac{|v - v'|}{\max\{r'(u), r'(u')\}}\right\}\right\}$$

where $M = M(v, v')$ is the unique solution to the equation $M = \frac{|v-v'|}{r'(M)}$. If $u = u' = 0$, and hence $\frac{|v-v'|}{\max\{r'(u),r'(u')\}}$ is undefined, we adopt the convention $d_r(w, w') = M$.

Remark 3. *From now on we simplify our notation by writing ℓ for ℓ_r, $d(w, w')$ for $d_r(w, w')$, $d_{CC}(w, w')$ for $d_{rCC}(w, w')$, Φ for Φ_r, and G for G_r. Most of what follows is true for all r-Grushin planes. We will clearly state when this is not the case and a result or example applies only to the classical Grushin plane where $r(u) = \frac{1}{2}u|u|$.*

The next lemma demonstrates that the Carnot-Carathéodory metric and the quasidistance on the r-Grushin plane are comparable.

Lemma 12. *There exists a positive constant C such that for any two points $w, w' \in G$*

$$\frac{1}{C}d_{CC}(w, w') \leq d(w, w') \leq C d_{CC}(w, w').$$

Proof. Let $w = (u, v)$ and $w' = (u', v')$ be points in G. We make use of the following facts:

(1) $d_{CC}((u, v), (u', v)) = |u - u'|$.

(2) $d_{CC}((u, v), (u, v')) \leq \frac{|v-v'|}{r'(u)}$ provided $u \neq 0$.

(1) is true since for all curves γ from (u, v) to (u', v), $\ell(\gamma) = \int_0^1 \sqrt{(\gamma_1'(s))^2 + \frac{(\gamma_2'(s))^2}{(r'(\gamma_1(s)))^2}}ds \geq \int_0^1 |\gamma_1'(s)|ds \geq |u - u'|$ and the length of the curve which is the horizontal line segment from (u, v) to (u', v) is $|u - u'|$. Similarly (2) is true since the length of the curve which is the vertical line segment from (u, v) to (u, v') is given by $\frac{|v-v'|}{r'(u)}$.

Now we show $\frac{1}{C}d_{CC}(w, w') \leq d(w, w')$. If $v = v'$, then $d(w, w') = |u - u'| = d_{CC}(w, w')$. Hence we may assume $v \neq v'$.

Case 1: $M \geq \frac{|v-v'|}{\max\{r'(u), r'(u')\}}$

By our convention we can assume either u or u' is nonzero. Without loss of generality we take $|u| \geq |u'|$ which gives us

$$\begin{aligned} d_{CC}(w,w')) &\leq d_{CC}((u,v),(u,v')) + d_{CC}((u,v'),(u',v')) \\ &\leq |u-u'| + \frac{|v-v'|}{r'(u)} \\ &\leq |u-u'| + \frac{|v-v'|}{\max\{r'(u), r'(u')\}} \\ &\leq 2\max\left\{|u-u'|, \frac{|v-v'|}{\max\{r'(u), r'(u')\}}\right\} \\ &= 2d(w,w'). \end{aligned}$$

Case 2: $M \leq \frac{|v-v'|}{\max\{r'(u), r'(u')\}}$

Then since by our definition $M = \frac{|v-v'|}{r'(M)}$, we have $\max\{r'(u), r'(u')\} \leq r'(M)$. Furthermore since r' is even and $r'|_{[0,\infty)}$ is increasing, we may conclude $\max\{|u|, |u'|\} \leq M$. Thus

$$\begin{aligned} d_{CC}(w,w') &\leq d_{CC}(w,(M,v)) + d_{CC}((M,v),(M,v')) + d_{CC}((M,v'),w') \\ &\leq |M-u| + |M-u'| + \frac{|v-v'|}{r'(M)} \\ &\leq 5M \\ &\leq 5d(w,w'). \end{aligned}$$

This proves $\frac{1}{C}d_{CC}(w,w') \leq d(w,w')$ for some constant C.

To prove $d(w,w') \leq Cd_{CC}(w,w')$ it suffices to show for an arbitrary path γ from w to w', $\ell(\gamma) \geq \frac{1}{2m}d(w,w')$. Recall m is the doubling constant defined in Lemma 11. We once again assume $|u| \geq |u'|$. Fix $\gamma = (\gamma_1, \gamma_2)$ and let s_0 be such that $|\gamma_1(s) - u| \leq |\gamma_1(s_0) - u|$ for all s. If $|\gamma_1(s_0) - u| \geq d(w,w')$, then since $\ell(\gamma) \geq |\gamma_1(s_0) - u|$, we have our desired inequality. Now we assume $|\gamma_1(s_0) - u| < d(w, w')$. In other words

$$|\gamma_1(s_0) - u| < \max\left\{|u - u'|, \min\left\{M, \frac{|v - v'|}{\max\{r'(u), r'(u')\}}\right\}\right\}.$$

By the definition of s_0, $|\gamma_1(s_0) - u| \geq |u - u'|$ and thus $|\gamma_1(s_0) - u| < M$. Also by the definition of s_0, $|\gamma_1(s)| \leq |u| + |\gamma_1(s_0) - u|$ for all s. Combining our inequalities gives $|\gamma_1(s)| < M + |u|$ for all s. Then

$$\begin{aligned}
\ell(\gamma) &= \int_0^1 \sqrt{(\gamma_1'(s))^2 + \frac{(\gamma_2'(s))^2}{(r'(\gamma_1(s)))^2}} ds \\
&\geq \int_0^1 \sqrt{(\gamma_1'(s))^2 + \frac{(\gamma_2'(s))^2}{(r'(|u| + M))^2}} ds \\
&\geq \frac{1}{2}\left(|u - u'| + \frac{|v - v'|}{r'(|u| + M)}\right) \\
&\geq \frac{1}{2}\left(|u - u'| + \frac{|v - v'|}{r'(2\max\{M, |u|\})}\right) \\
&\geq \frac{1}{2}\left(|u - u'| + \frac{|v - v'|}{mr'(\max\{M, |u|\})}\right) \text{ by Lemma 11} \\
&= \frac{1}{2}\left(|u - u'| + \frac{1}{m}\min\left\{\frac{|v - v'|}{r'(M)}, \frac{|v - v'|}{r'(u)}\right\}\right) \\
&\geq \frac{1}{2m} d(w, w') \text{ by the definition of } M.
\end{aligned}$$

□

We are now able to show that our quasidistance is equivalent to our Carnot-Carathéodory distance for the purpose of proving quasisymmetry.

Lemma 13. *If $w, a, b \in G$ are such that $d_{CC}(w, a) \leq d_{CC}(w, b)$, then $d(w, a) \leq C^2 d(w, b)$.*

Proof. By our previous lemma, $d(w, a) \leq C d_{CC}(w, a) \leq C d_{CC}(w, b) \leq C^2 d(w, b)$. □

6.3 The Quasisymmetric Equivalence of the Complex Plane and Generalized Grushin Planes

In this section we will prove two key lemmas which show how the quasidistance between two points in G_r compares to the distance between their images in the complex plane. These will allow us to finally prove the desired quasisymmetry with relative ease.

Recall the map $\Phi : G \to \mathbb{C}$ by

$$\Phi(u, v) = r(u) + iv. \tag{6.3.1}$$

We will eventually show Φ is a quasisymmetry. Throughout our proof we will use the sup norm on $\mathbb{C}$ so $|\Phi(w) - \Phi(w')| = |(r(u) - r(u'), v - v')| = \max\{|r(u) - r(u')|, |v - v'|\}$. The following two lemmas describe how $d(w, w')$ compares to $|\Phi(w) - \Phi(w')|$. Note the dependence on the relative magnitudes of $d(w, w')$ and the maximum distance of w and w' from the v-axis. This is unsurprising since the amount by which the metric on the Grushin plane is distorted from the Euclidean metric depends on a comparison between the same two quantities.

Lemma 14. *Suppose $w, w' \in G$ and $\max\{|u|, |u'|\} \geq d(w, w')$. Then for some constant C_1*

$$\frac{1}{C_1}|\Phi(w) - \Phi(w')| \leq d(w, w') \max\{r'(u), r'(u')\} \leq C_1|\Phi(w) - \Phi(w')|.$$

Proof. Fix w, w' such that $\max\{|u|, |u'|\} > d(w, w')$. Then $\max\{|u|, |u'|\} \geq |u - u'|$, and thus $uu' > 0$. By the mean value theorem and our conditions on r, for some c between u and u' we have

$$|r(u)-r(u')| = |u-u'|r'(c) \leq |u-u'| \max\{r'(u), r'(u')\} \leq |u-u'|\beta \max\left\{\frac{r(u)}{u}, \frac{r(u')}{u'}\right\} \leq \beta|r(u)-r(u')|. \tag{6.3.2}$$

The last inequality holds since our conditions on r imply the function $\frac{r(u)}{u}$ is increasing. Indeed, $\left(\frac{r(u)}{u}\right)' = \frac{ur'(u)-r(u)}{u^2} > 0$, because by definition $\frac{r(u)}{u} \leq r'(u)$.

If $M \leq \frac{|v-v'|}{\max\{r'(u),r'(u')\}}$, then $r'(M) \leq r'(d(w, w')) \leq \max\{r'(u), r'(u')\} \leq r'(M)$ which implies $r'(M) = \max\{r'(u), r'(u')\}$ and thus $M = \frac{|v-v'|}{\max\{r'(u),r'(u')\}}$. Therefore we may assume $M \geq \frac{|v-v'|}{\max\{r'(u),r'(u')\}}$ and $d(w, w') = \max\left\{|u - u'|, \frac{|v-v'|}{\max\{r'(u),r'(u')\}}\right\}$.

We obtain our result by considering the four cases given by the two choices for $|\Phi(w)-\Phi(w')|$ and the two choices for $d(w,w')$. If $|\Phi(w)-\Phi(w')| = |r(u)-r(u')|$ and $d(w,w') = |u-u'|$ then the result follows directly from (6.3.2). If $|\Phi(w)-\Phi(w')| = |r(u)-r(u')|$ and $d(w,w') = \frac{|v-v'|}{\max\{r'(u),r'(u')\}}$, then $|r(u)-r(u')| \le |u-u'|\max\{r'(u),r'(u')\} \le |v-v'| \le |r(u)-r(u')|$. If $|\Phi(w)-\Phi(w')| = |v-v'|$ and $d(w,w') = |u-u'|$, then $|v-v'| \le |u-u'|\max\{r'(u),r'(u')\} \le \beta|r(u)-r(u')| \le \beta|v-v'|$. Finally if $|\Phi(w)-\Phi(w')| = |v-v'|$ and $d(w,w') = \frac{|v-v'|}{\max\{r'(u),r'(u')\}}$, then $|v-v'| = d(w,w')\max\{r'(u),r'(u')\}$. □

Lemma 15. *Suppose* $w, w' \in G$ *and* $\max\{|u|,|u'|\} \le d(w,w')$. *Then for some constant* C_2,

$$\frac{1}{C_2}|\Phi(w)-\Phi(w')| \le r'(d(w,w'))d(w,w') \le C_2|\Phi(w)-\Phi(w')|.$$

Proof. Fix w, w' such that $\max\{|u|,|u'|\} \le d(w,w')$. If $d(w,w') = \frac{|v-v'|}{\max\{r'(u),r'(u')\}}$, then

$$r'(M) \le \max\{r'(u),r'(u')\} \le r'(d(w,w')) = r'\left(\frac{|v-v'|}{\max\{r'(u),r'(u')\}}\right) \le r'(M)$$

which implies $M = \frac{|v-v'|}{\max\{r'(u),r'(u')\}}$. Thus $d(w,w') = \max\{|u-u'|, M\}$.

We also have

$$\frac{1}{2}|r(u)-r(u')| \le \max\{|r(u)|,|r(u')|\} \le \max\{r'(u)|u|, r'(u')|u'|\} \le r'(d(w,w'))d(w,w'). \quad (6.3.3)$$

Furthermore by our hypothesis, if $d(w,w') = |u-u'|$, we must have $uu' \le 0$ which implies

$$\begin{aligned} r'(d(w,w'))d(w,w') &= r'(u-u')|u-u'| \\ &\le r'(2\max\{|u|,|u'|\})|u-u'| \\ &\le m\max\{r'(u),r'(u')\}|u-u'| \text{ by Lemma 11} \\ &\le m(r'(u)+r'(u'))|u-u'| \\ &= m|ur'(u)-u'r'(u')+ur'(u')-u'r'(u)| \\ &\le 2m|ur'(u)-u'r'(u')| \end{aligned}$$

$$\leq \quad 2m\beta|r(u)-r(u')|$$

where the last inequality holds because $uu' \leq 0$. We once again complete our proof by considering the four cases given by the two choices for $|\Phi(w)-\Phi(w')|$ and the two choices for $d(w,w')$. If $|\Phi(w)-\Phi(w')| = |r(u)-r(u')|$ and $d(w,w') = |u-u'|$ then the result follows directly from (6.3.3) and (6.3.4). If $|\Phi(w)-\Phi(w')| = |r(u)-r(u')|$ and $d(w,w') = M$, then $\frac{1}{2}|r(u)-r(u')| \leq r'(d(w,w'))d(w,w') = Mr'(M) = |v-v'| \leq |r(u)-r(u')|$. If $|\Phi(w)-\Phi(w')| = |v-v'|$ and $d(w,w') = |u-u'|$, then $|v-v'| = Mr'(M) \leq r'(u-u')|u-u'| \leq 2m\beta|r(u)-r(u')| \leq 2m\beta|v-v'|$. Finally if $|\Phi(w)-\Phi(w')| = |v-v'|$ and $d(w,w') = M$, then $|v-v'| = r'(d(w,w'))d(w,w')$. $\square$

Now we are able to show Φ is a quasisymmetry. We actually only prove weak quasisymmetry, but this is equivalent to quasisymmetry for the spaces we are considering. For the definition of weak quasisymmetry and a discussion of why proving weak quasisymmetry for the Grushin plane suffices please see 3.2.

Definition 12. *Let (X,d_X) and (Y,d_Y) be metric spaces. A homeomorphism $\Phi : X \to Y$ is weakly quasisymmetric if there exists a constant C such that for all triples of points $a,b,c \in X$ we have*

$$d_X(a,b) \leq d_X(b,c) \implies d_Y(\Phi(a),\Phi(b)) \leq C d_Y(\Phi(b),\Phi(c))$$

For a proof of the equivalence of weak quasisymmetry and quasisymmetry the reader is referred to Theorem 10.15 in [7].

Theorem 9. *Suppose a, b and w are points in the r-Grushin plane such that $d_{CC}(w,a) \leq d_{CC}(w,b)$. Then for some constant $C(r)$ we have*

$$|\Phi(w)-\Phi(a)| \leq C(r)|\Phi(w)-\Phi(b)|.$$

Proof. Fix $a,b,w \in G$ such that $d_{CC}(w,a) \leq d_{CC}(w,b)$. Then $d(w,a) \leq C^2 d(w,b)$ by Lemma 13.

We divide the proof into the following four cases:

Case 1: $\max\{|u|,|a_1|\} \leq d(w,a)$ **and** $\max\{|u|,|b_1|\} \leq d(w,b)$

By Lemma 15,

$$|\Phi(w)-\Phi(a)| \leq C_2 r'(d(w,a))d(w,a) \leq C_2 C^2 r'(C^2 d(w,b))d(w,b) \leq C'|\Phi(w)-\Phi(b)|$$

where C' is such that $C_2^2 C^2 r'(C^2 t) \leq C' r'(t)$. Such a C' can be found since r is doubling.

Case 2: $\max\{|u|,|a_1|\} \leq d(w,a)$ **and** $\max\{|u|,|b_1|\} \geq d(w,b)$

This case is the same as Case 1 except one should use Lemma 14 instead of Lemma 15 at the end of the chain of inequalities. More specifically we have $|\Phi(w)-\Phi(a)| \leq C_2 r'(d(w,a))d(w,a) \leq C_2 C^2 r'(C^2 d(w,b))d(w,b) \leq C_2 C^2 r'(C^2 \max\{|u|,|b_1|\})d(w,b) \leq \frac{C''}{C_1} d(w,b) \max\{r'(u), r'(b_1)\} \leq C''|\Phi(w)-\Phi(b)|$ where C'' is such that $C_2 C^2 r'(C^2 t)) \leq \frac{C''}{C_1} r'(t)$.

The last two cases are slightly more complicated since first we must find ways to compare $\max\{|u|,|a_1|\}$ with $\max\{|b_1|,|u|\}$ and $d(w,b)$. After these inequalities are obtained, the proofs follow similarly to those of the first two cases.

Case 3: $\max\{|u|,|a_1|\} \geq d(w,a)$ **and** $\max\{|u|,|b_1|\} \geq d(w,b)$

Since $d(w,a) \leq C^2 d(w,b)$, we have $|a_1 - u| \leq C^2 d(w,b)$ and therefore $|a_1| \leq |u| + C^2 d(w,b)$. Then we can obtain our desired comparison:

$$\max\{|u|,|a_1|\} \leq \max\{|u|,|b_1|\} + C^2 d(w,b) \leq (1+C^2)\max\{|u|,|b_1|\}.$$

Finally we have

$$\begin{aligned} |\Phi(w)-\Phi(a)| &\leq C_1 d(w,a)\max\{r'(u), r'(a_1)\} \\ &\leq C_1 C^2 d(w,b) r'((1+C^2)\max\{|b_1|,|u|\}) \end{aligned}$$

$$\leq \quad C'''|\Phi(w) - \Phi(b)|$$

where C''' is such that $C_1^2 C^2 r'((1+C^2)t) \leq C''' r'(t)$.

Case 4: $\max\{|u|, |a_1|\} \geq d(w,a)$ **and** $\max\{|u|, |b_1|\} \leq d(w,b)$

Similarly to the previous case $d(w,a) \leq C^2 d(w,b)$ implies $|a_1 - u| \leq C^2 d(w,b)$ and therefore $|a_1| \leq |u| + C^2 d(w,b)$. Then we have

$$\max\{|u|, |a_1|\} \leq \max\{|u|, |b_1|\} + C^2 d(w,b) \leq (1+C^2) d(w,b).$$

Thus

$$\begin{aligned} |\Phi(w) - \Phi(a)| &\leq C_1 d(w,a) \max\{r'(u), r'(a_1)\} \\ &\leq C_1 C^2 r'((1+C^2) d(w,b)) d(w,b) \\ &\leq C''''|\Phi(w) - \Phi(b)| \end{aligned}$$

where C''' is such that $C_1 C_2 C^2 r'((1+C^2)t) \leq C'''' r'(t)$. □

Since we have shown that the r-Grushin plane is quasisymmetrically equivalent to $\mathbb{C}$, we may ask whether all of our restrictions on the homeomorphism r were necessary. The requirement that r' is even can almost certainly be eliminated, since it is mostly used to simplify the proof when dealing with w and w' on opposite sides of the v-axis. The following theorem demonstrates that the other major constraint on r is a necessary condition.

Theorem 10. *Let $r : \mathbb{R} \to \mathbb{R}$ be a differentiable homeomorphism such that $r'|_{[0,\infty)}$ and $r'|_{(-\infty,0]}$ are both homeomorphisms onto $[0,\infty)$, $r(0) = 0$, and Φ, as defined in (6.3.1) is quasisymmetric. Then there exists $\beta > 1$ such that for all $u \in \mathbb{R} \setminus \{0\}$*

$$\frac{r(u)}{u} \leq r'(u) \leq \beta \frac{r(u)}{u}.$$

Proof. If u is positive, by the mean value theorem there exists $c \in (0,u)$ such that $r'(c) = \frac{r(u)}{u}$. Then since r' is a homeomorphism of $[0,\infty)$ and is therefore increasing on $[0,\infty)$, we have $r'(u) > r'(c)$. Thus $r'(u) \geq \frac{r(u)}{u}$. To achieve an upper bound we again use the mean value theorem except this time on the interval $[u, 2u]$. This gives

$$r'(u) \leq \frac{r(2u)-r(u)}{u} \leq \beta\frac{r(u)-r(0)}{u} = \frac{\beta r(u)}{u}.$$

The second inequality holds since Φ is quasisymmetric, and as stated in the proof of Lemma 12, we have $d((u,v),(u',v)) = |u-u'|$. Hence there exists some $\beta > 1$ such that $r(2u)-r(u) \leq \beta(r(u)-r(0))$.

Now we assume u is negative. By the mean value theorem there exists $c \in (u,0)$ such that $r'(c) = \frac{r(u)}{u}$. Then since r' is a homeomorphism of $(-\infty,0]$ onto $[0,\infty)$ and is therefore decreasing on $(-\infty,0)$, we have $r'(u) > r'(c)$. Thus $r'(u) \geq \frac{r(u)}{u}$. To achieve an upper bound we use the mean value theorem on the interval $[2u, u]$. This gives

$$r'(u) \leq \frac{r(2u)-r(u)}{u} \leq \beta\frac{r(u)-r(0)}{u} = \frac{\beta r(u)}{u}.$$

The second inequality holds once again since Φ is quasisymmetric, and as stated in the proof of Lemma 12, we have $d((u,v),(u',v)) = |u-u'|$. Hence there exists some $\beta > 1$ such that $r(2u)-r(u) \leq \beta(r(u)-r(0))$. □

6.4 An Analytic Definition of Quasisymmetry

In this section we will use conjugation by our quasisymmetry Φ to develop an analytic definition of quasisymmetry in the r-Grushin plane.

For the next several results let $g = (g_1, g_2) : G \to G$ be a homeomorphism. We define $f = f_1 + if_2 : \mathbb{C} \to \mathbb{C}$ to be the conjugation of g by Φ. In other words $f = \Phi \circ g \circ \Phi^{-1}$. Let $U = \frac{\partial}{\partial u}$ and $V = r'(u)\frac{\partial}{\partial v}$ be the vector fields corresponding to our metric on the r-Grushin plane. Recall in Definition 9 we gave the notation $W = \frac{1}{2}(U - iV)$, $\overline{W} = \frac{1}{2}(U + iV)$ and $\tilde{g} = \Phi \circ g$.

The next lemma demonstrates a relationship between the classical Beltrami equation and the r-Grushin Beltrami equation both of which were defined in the introduction. The following theorem is an analytic definition of quasisymmetry on the r-Grushin plane.

Lemma 16. *Suppose f and g have partial derivatives that exist almost everywhere. Then $\tilde{g}$ satisfies the r-Grushin Beltrami equation if and only if f satisfies the classical Beltrami equation. The equations are stated in Definitions 3 and 9 respectively.*

Proof. Wherever our derivatives exist we have by the chain rule:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}|_{\Phi(w)} & \frac{\partial f_1}{\partial y}|_{\Phi(w)} \\ \frac{\partial f_2}{\partial x}|_{\Phi(w)} & \frac{\partial f_2}{\partial y}|_{\Phi(w)} \end{pmatrix} \begin{pmatrix} \frac{\partial \Phi_1}{\partial u}|_{w} & \frac{\partial \Phi_1}{\partial v}|_{w} \\ \frac{\partial \Phi_2}{\partial u}|_{w} & \frac{\partial \Phi_2}{\partial v}|_{w} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial u}|_{g(w)} & \frac{\partial \Phi_1}{\partial v}|_{g(w)} \\ \frac{\partial \Phi_2}{\partial u}|_{g(w)} & \frac{\partial \Phi_2}{\partial v}|_{g(w)} \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial u}|_{w} & \frac{\partial g_1}{\partial v}|_{w} \\ \frac{\partial g_2}{\partial u}|_{w} & \frac{\partial g_2}{\partial v}|_{w} \end{pmatrix}$$

which implies

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}|_{\Phi(w)} & \frac{\partial f_1}{\partial y}|_{\Phi(w)} \\ \frac{\partial f_2}{\partial x}|_{\Phi(w)} & \frac{\partial f_2}{\partial y}|_{\Phi(w)} \end{pmatrix} \begin{pmatrix} r'(u) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r'(g_1) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial u}|_{w} & \frac{\partial g_1}{\partial v}|_{w} \\ \frac{\partial g_2}{\partial u}|_{w} & \frac{\partial g_2}{\partial v}|_{w} \end{pmatrix}$$

and thus

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}|_{\Phi(w)} & \frac{\partial f_1}{\partial y}|_{\Phi(w)} \\ \frac{\partial f_2}{\partial x}|_{\Phi(w)} & \frac{\partial f_2}{\partial y}|_{\Phi(w)} \end{pmatrix} = \begin{pmatrix} \frac{1}{r'(u)} U(r(g_1))|_w & \frac{1}{r'(u)} V(r(g_1))|_w \\ \frac{1}{r'(u)} U(g_2)|_w & \frac{1}{r'(u)} V(g_2)|_w \end{pmatrix}. \tag{6.4.1}$$

Therefore

$$\mu \circ \Phi = \frac{\frac{\partial f}{\partial \bar{z}}}{\frac{\partial f}{\partial z}} = \frac{U(r(g_1)) - V(g_2) + i(U(g_2) + V(r(g_1)))}{U(r(g_1)) + V(g_2) + i(U(g_2) - V(r(g_1)))} = \frac{\overline{W}\tilde{g}}{W\tilde{g}} \text{ a.e.}$$

□

We require a definition of absolute continuity on lines in the r-Grushin plane before giving our theorem.

Definition 13. *Suppose g is a homeomorphism of the r-Grushin plane. We say $\tilde{g}$ is absolutely continuous on a horizontal interval $I_v = \{(u, v) : a < u < b\}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\{[w_i, w_i']\}_{1 \le i \le n}$ is a disjoint collection of sub-intervals of I_v*

$$\Sigma_{i=1}^n |\Phi(w_i') - \Phi(w_i)| < \delta \implies \Sigma_{i=1}^n |\tilde{g}(w_i') - \tilde{g}(w_i)| < \epsilon.$$

We define absolute continuity on vertical line segments analogously.

The function $\tilde{g}$ is absolutely continuous on lines if for every rectangle $R = \{(u, v) : a < u < b, c < v < d\}$, $\tilde{g}$ is absolutely continuous on a.e. horizontal interval $I_v = \{(u, v) : a < u < b\}$ and a.e. vertical interval $I_u = \{(u, v) : c < v < d\}$ where almost every is with respect to Lebesgue measure.

We have defined absolute continuity on lines in this manner so that f is absolutely continuous on lines exactly when $\tilde{g}$ is absolutely continuous on lines. To see why this is true replace w_i' with $\Phi^{-1}(z_i')$ and w_i with $\Phi^{-1}(z_i)$ in the definition above, and recall Φ maps vertical and horizontal intervals to vertical and horizontal intervals respectively.

We now prove Theorem 8.

Proof. Suppose g is quasisymmetric. Then since Φ is quasisymmetric, it follows that f is quasisymmetric and hence quasiconformal. So by the analytic definition of quasiconformality, the partial derivatives of f exist almost everywhere and where they exist

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$$

for some measurable μ with $||\mu||_\infty < 1$. Furthermore f is absolutely continuous on lines, which implies $\tilde{g}$ is absolutely continuous on lines. Since each component of Φ and Φ^{-1} is differentiable except at the vertical axis, the partial derivatives of g exist a.e. Therefore $\overline{W}\tilde{g}$ and $W\tilde{g}$ exist a.e. and by our lemma

$$\overline{W}\tilde{g} = (\mu \circ \Phi) W\tilde{g}.$$

Since $||\mu||_\infty < 1$ we have $||\nu||_\infty = ||\mu \circ \Phi||_\infty < 1$.

Now suppose $\tilde{g}$ is absolutely continuous on lines and satisfies equation (6.1.1) for all points at which it is defined. Then f is absolutely continuous on lines and hence has partial derivatives that exist a.e. As in our proof of the forwards implication, this implies that the partial derivatives of g exist a.e., and g satisfies the r-Grushin Beltrami equation. Therefore f satisfies the classical Beltrami equation and is thus quasiconformal. Finally by conjugation, g is quasisymmetric. $\square$

One would like to be able to replace quasisymmetry with quasiconformality in this theorem. It is a well known result that quasisymmetry implies quasiconformality [7]. However, the converse does not always occur, and so far we have been unable to either prove or disprove it for the r-Grushin plane. A partial answer to our question is in Theorem 11, where we will show that on certain domains ν being identically zero implies g is conformal. The limitations on the domain arise when g does not preserve the singular line. We will discuss this following Theorem 11.

6.5 Conformal Mappings on the r-Grushin Planes

Since conformal mappings play a vital role in the study of quasiconformal mappings, it is of interest to us to find a useful characterization of them on the r-Grushin plane. We will first develop a definition of conformality on the r-Grushin plane from the definition of conformal mappings on Riemannian manifolds. This is appropriate since the r-Grushin plane is Riemannian everywhere except on the singular line. Throughout the rest of the section we will provide further justification for our definition by looking at the classical Beltrami definition of conformality, and an earlier paper by Payne [20].

Let M be a C^∞ Riemannian manifold and g be a homeomorphism from M to M. Recall g is conformal if the pullback of the Riemannian metric by g is equal to the metric multiplied by some positive function. Since we assume M is C^∞, we also have g is infinitely differentiable [14]. The length element for our metric on $G \setminus \{u = 0\}$ is

$$du^2 + \frac{dv^2}{(r'(u))^2}$$

and its pullback by a function $g : G \to G$ is

$$\left[(U(g_1))^2 + \frac{(U(g_2))^2}{(r'(g_1))^2}\right] du^2 + \frac{1}{(r'(u))^2}\left[(V(g_1))^2 + \frac{(V(g_2))^2}{(r'(g_1))^2}\right] dv^2 + \frac{2}{r'(u)}\left[U(g_1)V(g_1) + \frac{U(g_2)V(g_2)}{(r'(g_1))^2}\right] dudv.$$

Recall r' is zero only at zero so these expressions make sense on $G \setminus \{u = 0\}$ whenever g_1 is also non-zero on this domain. Then for g to be conformal in the sense described above we must have

$$\frac{2}{r'(u)}\left[U(g_1)V(g_1)+\frac{U(g_2)V(g_2)}{(r'(g_1))^2}\right]=0$$

and

$$\frac{1}{(r'(u))^2}\left[(U(g_1))^2+\frac{(U(g_2))^2}{(r'(g_1))^2}\right]=\frac{1}{(r'(u))^2}\left[(V(g_1))^2+\frac{(V(g_2))^2}{(r'(g_1))^2}\right].$$

Hence

$$V(g_1)=\frac{1}{r'(g_1)}U(g_2) \text{ and } U(g_1)=-\frac{1}{r'(g_1)}V(g_2).$$

Thus we have found a version of the Cauchy-Riemann equations for the r-Grushin plane. We define conformality on the r-Grushin plane as follows:

Definition 14. *Suppose A and B are domains in G and $g=(g_1,g_2): A\to B$ is a homeomorphism. Define $A'=A\setminus\{u=0 \text{ or } g_1(u)=0\}$. We say $g|_{A'}$ is conformal provided that*

$$D_r g=\begin{pmatrix} U(g_1) & V(g_1) \\ \frac{U(g_2)}{r'(g_1)} & \frac{V(g_2)}{r'(g_1)} \end{pmatrix}$$

is defined and is a conformal matrix for every point in A'. We say g is conformal on all of A if g is conformal on A' and for all points $w_0\in A\setminus A'$,

$$\lim_{w\to w_0} D_r g(w)$$

is defined and non-zero. We take the limit along all paths in A'.

At first it may be tempting to think that the conjugation by Φ of any conformal map in the complex plane should be a conformal map in the r-Grushin plane. This is not quite true. There are mappings that are conformal everywhere on the complex plane, but when conjugated by Φ are only conformal on domains limited by the singular line. For example, consider a horizontal translation $f(x+iy)=x+a+iy$. When we conjugate f with $\Phi_c(u,v)=r_c(u)+iv$ where $r_c(u)=\frac{1}{2}u|u|$ we obtain the mapping

$$g(u,v) = \left(\sqrt{2}\frac{\frac{1}{2}u|u| + a}{\sqrt{|\frac{1}{2}u|u| + a|}}, v \right).$$

Notice $V(g_1) = \frac{U(g_2)}{r_c'(g_1)} = 0$ and $\frac{V(g_2)}{r_c'(g_1)} = U(g_1) = \frac{|u|}{\sqrt{|u|u|+2a|}}$, and thus $D_{r_c}g$ is singular exactly on the line $u = 0$, and the pre-image under g of the line $u = 0$. Therefore g is only conformal on the Grushin plane on a domain excluding the singular line and the pre-image of the singular line. We will discuss what must happen for a homeomorphism to be conformal on the entire Grushin plane after the next theorem.

The following result shows that for most domains in G our description of conformality matches with the classical Beltrami differential definition of conformality.

Theorem 11. *Suppose A and B are domains in G, and $g : A \to B$ is an orientation-preserving homeomorphism. Then g is conformal on the domain $A' = A \setminus \{(u,v) : u = 0$ or $g_1(u,v) = 0\}$ if and only if g is quasisymmetric and the Beltrami differential ν is identically zero on A'.*

Proof. Suppose g is conformal. Then all the derivatives in the entries of $D_r g$ must exist. Thus since g is orientation-preserving and $D_r g$ is conformal, we must have

(1) $U(r(g_1)) = V(g_2)$ and $V(r(g_1)) = -U(g_2)$, and

(2) $D_r g$ is non-singular.

Then by condition (1), $\overline{W}\tilde{g} = 0$ which implies $\nu = 0$ and by Lemma 16, $\mu = 0$. So we have $\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}$ and $\frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}$. Furthermore

$$J(D_r g) = \frac{1}{r'(g_1)}(U(g_1)V(g_2) - U(g_2)V(g_1)) \text{ and } J(Df) = \frac{r'(g_1)}{(r'(u))^2}(U(g_1)V(g_2) - U(g_2)V(g_1))|_{\Phi^{-1}(x,y)}.$$

The formula for $J(Df)$ can be easily computed from equation (6.4.1). Thus since r' takes the value zero only at $u = 0$, $J(D_r g)$ and $J(Df) \circ \Phi$ are zero for the exact same values on A'. So Df is conformal almost everywhere and therefore f is conformal and hence quasisymmetric. Finally, since compositions of quasisymmetric maps are quasisymmetric, g is quasisymmetric.

Now we assume g is quasisymmetric and ν is identically zero on A'. Since ν is identically zero on A', we must have $\overline{W}\tilde{g} = 0$, which implies condition (1). Also since g is quasisymmetric, f is quasisymmetric and hence quasiconformal. By Lemma 16, $\mu = 0$ and hence f is conformal. Thus we can conclude that condition (2) also holds by our earlier statement regarding the Jacobians of $D_r g$ and Df, and therefore g is conformal. $\square$

The situation is more complicated if we include the singular line and its pre-image in our domain. For g to be conformal in such a domain,

$$J(D_r g) = \frac{r'(u)}{r'(g_1)} \left(\frac{\partial g_1}{\partial u} \frac{\partial g_2}{\partial v} - \frac{\partial g_2}{\partial u} \frac{\partial g_1}{\partial v} \right)$$

must be non-singular. Since we assume g is orientation-preserving, this occurs exactly when

$$\lim_{u \to 0} \frac{r'(u)}{r'(g_1)}$$

is finite and non-zero. Thus the singular line must map to itself.

This theorem also justifies our earlier work and in particular our selection of a relationship between the quasisymmetry Φ and the vector fields on G. With other choices we do not have that $\overline{W}\tilde{g} = 0$ when g is conformal. For example, if we use Meyerson's quasisymmetry

$$\Phi_M(u, v) = r_M(u) + iv = u|u| + iv$$

for the classical Grushin plane with vector fields $U = \frac{\partial}{\partial u}$ and $V = |u| \frac{\partial}{\partial v}$ we do not have that $|u|$ is equal $r'_M(u)$. We compute

$$\overline{W}\tilde{g} = \frac{1}{2}(U(g_1|g_1|) - V(g_2) + i(U(g_2) + V(g_1|g_1|))).$$

Also we can use the same method as described at the beginning of this section to say, since g is a homeomorphism on the classical Grushin plane, g is conformal exactly when

$$\begin{pmatrix} U(g_1) & V(g_1) \\ \frac{U(g_2)}{|g_1|} & \frac{V(g_2)}{|g_1|} \end{pmatrix}$$

is a conformal matrix. To simplify matters for the moment, we assume our domain does not include points on the singular line or points that map to the singular line. Thus if g is conformal, we must have $|g_1|U(g_1) = V(g_2)$ and $|g_1|V(g_1) = -U(g_2)$ which implies $U(g_1|g_1|) = 2V(g_2)$ and $V(g_1|g_1|) = -2U(g_2)$. Hence we are not guaranteed that $\overline{W}\tilde{g} = 0$ for conformal mappings.

To the best of the author's knowledge the only earlier discussion of conformal mappings on the Grushin plane is in a paper by Payne [20]. He defines a sequence of flows and states that the time-s maps induced by the solutions to any of the flows are conformal maps on the Grushin plane. Here we will look at a generalization of Payne's flows and show that their solutions induce conformal maps on the r-Grushin plane. In the following calculations x and y will be formal variables and u and v will be the Grushin coordinates as before. First we define a sequence of functions of x and y, $(\xi_k(x,y), \eta_k(x,y))$, $k \in \mathbb{N}$ by $(\xi_1, \eta_1) = (0, 1)$,

$$(\xi_2, \eta_2) = \left(\frac{r(x)}{r'(x)}, y\right),$$

and the functions given inductively by

$$(\xi_k, \eta_k) = (2\xi_{k-1}\eta_{k-1}, \eta_{k-1}^2 - (r'(x)\xi_{k-1})^2) \text{ for } k \geq 3.$$

The flows we will be solving are the autonomous differential equations:

$$\left(\frac{\partial x_k}{\partial s}, \frac{\partial y_k}{\partial s}\right) = (\xi_k(x_k, y_k), \eta_k(x_k, y_k))$$

where $x_k = x_k(s, u, v)$ and $y_k = y_k(s, u, v)$ are functions of u, v and a time parameter s. We will let g_k denote (x_k, y_k), In other words $g_k = (x_k, y_k) : [0, \infty)$ x $G \to G$. When $r = \frac{1}{2}u|u|$, these flows agree with Payne's flows up to a normalization. We will show that each time-s map associated with a solution with initial condition $x_k(0,u,v) = u$ and $y_k(0,u,v) = v$, is a conformal map on some domain in the r-Grushin plane.

One can easily compute the solutions to the first two flows $g_1 = (u, v+s)$, and $g_2 = (r^{-1}(r(u)e^s), ve^s)$, and check that the time-s maps satisfy our definition of conformality. The first solution gives vertical shifts by s. In the classical Grushin plane ($r = \frac{1}{2}u|u|$) the second solution gives dilations by a factor of $e^{s/2}$.

To solve the remaining equations we will use the following auxiliary functions:

$$\Phi(x,y) = r(x) + iy \text{ and } b_k(x,y) = r'(x)\xi_k(x,y) + i\eta_k(x,y).$$

Recall x and y are formal variables. We are interested in b_k because

$$b_k \circ g_k = \frac{\partial}{\partial s}(\Phi \circ g_k). \tag{6.5.1}$$

We will then find a non-iterative way of expressing $b_k(x,y)$ for each k value and finally integrate $b_k \circ g_k$ to solve for $\Phi \circ g_k$. We choose to solve for $\Phi \circ g_k$ instead of solving for g_k directly, because this is a far easier task as will be evident when the reader sees the solutions in a moment. We use the definitions of b_k, ξ_k and η_k to compute

$$\begin{aligned}
b_k(x,y) &= r'(x)\xi_k(x,y) + i\eta_k(x,y) \\
&= r'(x)(2\xi_{k-1}\eta_{k-1}) + i(\eta_{k-1}^2 - (r'(x)\xi_{k-1})^2) \\
&= -i[2r'(x)\xi_{k-1}\eta_{k-1}i + (r'(x)\xi_{k-1})^2 - \eta_{k-1}^2] \\
&= -i[r'(x)\xi_{k-1}(x,y) + i\eta_{k-1}]^2 \\
&= -i(b_{k-1}(x,y))^2 \text{ for } k \geq 4
\end{aligned}$$

and

$$\begin{aligned}
b_3(x,y) &= r'(x)\xi_3(x,y) + i\eta_3(x,y) \\
&= r'(x)(2\xi_2\eta_2) + i(\eta_2^2 - (r'(x)\xi_2)^2)
\end{aligned}$$

$$\begin{aligned} &= r'(x)\left(2\frac{r(x)}{r'(x)}y\right) + i\left(y^2 - \left(r'(x)\frac{r(x)}{r'(x)}\right)^2\right) \\ &= 2r(x)y + i(y^2 - (r(x))^2) \\ &= -i[2r(x)yi + (r(x))^2 - y^2] \\ &= -i[r(x) + iy]^2 \\ &= -i\Phi(x,y)^2. \end{aligned}$$

Thus by induction we obtain

$$b_k(x,y) = i(i\Phi(x,y))^\alpha$$

where $\alpha = 2^{k-2}$ and $k \geq 3$. Then by equation (6.5.1) we have the following differential equations

$$\frac{\partial}{\partial s}(\Phi \circ g_k) = i(i(\Phi \circ g_k))^\alpha.$$

Thus

$$\frac{1}{1-\alpha}(\Phi \circ g_k)^{1-\alpha} = i(i)^\alpha s + C.$$

Recall our initial condition on $g_k = (x_k, y_k)$ was $x_k(0,u,v) = u$ and $y_k(0,u,v) = v$. So our initial condition is now $\Phi(x_k(0,u,v), y_k(0,u,v)) = r(u) + iv$ and thus $C = \frac{1}{1-\alpha}(r(u)+iv)^{1-\alpha}$. Finally we obtain the solutions

$$\Phi \circ g_k(s,u,v) = \frac{r(u) + iv}{([1-\alpha][-i(r(u)+iv)]^{\alpha-1}s + 1)^{\frac{1}{\alpha-1}}}.$$

Let $g_k^s : G \to G$ denote the map g_k for some fixed time s. We will show for all $k \in \mathbb{N}$ and all $s \in [0,\infty)$, g_k^s is conformal on some domain in the r-Grushin plane. For $k \in \{1,2\}$, g_k^s is conformal on the entire plane. For $k \geq 4$, g_k^s is conformal on some domain limited by a branch cut. For example when $k = 4$ if we specify that $-3i(r(u)+iv)^3 s + 1 \in \mathbb{C} \setminus \{(u,v) : u \leq 0, v = 0\}$ then we find that g_4^s is conformal on the domain

$$G - \left\{\Phi^{-1}(z) : \arg(z) \in \left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}, |z| > \left(\frac{1}{3s}\right)^{1/3}\right\}.$$

In general g_k^s will be conformal on a domain with $\alpha - 1$ cuts when $k \geq 4$. We will discuss the case of $k = 3$ after we prove conformality.

To prove each g_k^s for $k \geq 3$ is conformal we look at the function

$$f_k^s = \Phi \circ g_k^s \circ \Phi^{-1}(z) = \frac{z}{([1-\alpha][-iz]^{\alpha-1}s + 1)^{\frac{1}{\alpha-1}}}.$$

Thus

$$\frac{\partial f_k^s}{\partial \bar{z}} = 0,$$

and hence f_k^s is conformal. Then by Lemma 16, Theorem 8 and Theorem 11, g_k^s is conformal.

Earlier we noted that in the classical Grushin plane g_1^s and g_2^s were the familiar Grushin translations and dilations. Now we can see that g_3^s comes from a composition of translations, dilations and an inversion. We have

$$f_3^s = \Phi \circ g_3^s \circ \Phi^{-1} = \frac{z}{1 + izs} = \overline{\lambda^{-1}} \circ \ I_E \circ \lambda$$

where $\lambda(z) = zs - i$, $\Phi(u, v) = \frac{1}{2}u|u| + iv$ and I_E is the Euclidean inversion $z \to 1/\bar{z}$.

The family of maps generated by f_3 is not entirely satisfactory since as s goes to infinity f_3^s degenerates to the zero map. The slightly different family of maps $f_3^*(z) = \frac{is+z}{1+isz}$, goes to an inversion map as s goes to infinity which is the behavior we would expect.

A natural question to ask at this point is whether Payne's family of conformal maps includes all conformal maps on the Grushin plane or in some way generates all conformal maps on the Grushin plane. If this is not true, are there functions we could add to Payne's list to enable us to obtain all conformal maps? One way to approach these questions would be to try to find a more complete version of Theorem 11. In other words try to determine under exactly which conditions the theorem holds when we allow our domain to include the singular line. This would give a complete characterization of conformal mappings on the Grushin plane which then could be compared to Payne's class of maps.

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